INC 341 Feedback Control Systems: Lecture 2 Transfer Function of Dynamic Systems I

Asst. Prof. Dr.-Ing. Sudchai Boonto

Department of Control Systems and Instrumentation Engineering
King Mongkut's University of Technology Thonburi

Learning Outcomes

After finishing this lecture, the student should be able to:

1. Find the Laplace transform of time functions and the inverse Laplace transform.
2. Find the transfer function from a ODE and solve the ODE using the transfer function.
3. Find the transfer function from LTI electrical networks.
4. Find the transfer function from LTI translational mechanical systems.
5. Find the transfer function from LTI rotational mechanical systems.

Response by Convolution

There are two analytical techniques can be applied to Linear time-invariant systems (LTIs):

- the principle of superposition
- Using the convolution of the input with the unit impulse response of the system to determine the response of LTI systems.

Superposition

The homogeneity and additivity properties together are called the superposition principle. A linear function is one that satisfies the properties of superposition. Which is defined as

$$\begin{aligned} f(x_1 + x_2) = f(x_1) + f(x_2) = f_1 + f_2, &\qquad \text{Additivity} \\ f(ax) = af(x) \quad \forall a \in \mathbb{R}, &\qquad \text{Homogeneity} \end{aligned}$$

Total response

The total response of the LTI system with all zero initial conditions is

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = u(t) * h(t),$$

where$h(\tau)$ is the impulse response and $u(t)$ is the system input.

Superposition Example:
Consider the first order system

$$\dot{y} + ky = u$$

Show that the superposition holds for the system.
Let y_1 and y_2 are the response to the inputs u_1 and u_2. Multiply the first response with α_1 and the second with α_2, we have

$$\begin{aligned} k_1\dot{y}_1 + \alpha_1 k y_1 &= \alpha_1 u_1 \\ k_2\dot{y}_2 + \alpha_2 k y_2 &= \alpha_2 u_2 \end{aligned}$$

Adding them yields

$$\frac{d}{dt}\left[\alpha_1 y_1 + \alpha_2 y_2\right] + k\left[\alpha_1 y_1 + \alpha_2 y_2\right] = \alpha_1 u_1 + \alpha_2 u_2$$

Therefor, when the input is $k_1u_1 + k_2u_2$, the system response is $k_1y_1 + k_2y_2$.
Consequently, the system satisfy the superposition.

Convolution Example:
Consider the impulse response of the first-order system

$$h(t) = e^{-kt}\mathbb{1}(t)$$

The response to a general input is given by the convolution of the impulse response and the input:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{-k\tau}\mathbb{1}(t)u(t-\tau)d\tau \\ &= \int_{0}^{\infty} e^{-k\tau}u(t-\tau)d\tau \end{aligned}$$

If $u(t)$ is also a unit-step signal, we have

$$y(t) = \int_0^{\infty} e^{-k\tau}\mathbb{1}(t-\tau)d\tau = \int_0^{t} e^{-k\tau}d\tau = \frac{1}{k}\left[1 - e^{-kt}\right], \quad t \geq 0$$

Laplace Transform Review

The (unilateral) Laplace transform is defined as

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt,$$

where $s = \sigma + j\omega$, a complex variable.

The inverse Laplace transform is defined as

$$\mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$
$$= f(t)u(t)$$

Example:

$$\begin{aligned}\mathcal{L}\left[\mathbb{1}(t)\right] &= \int_{0^-}^{\infty} \mathbb{1}(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_{0^-}^{\infty} \\ &= \frac{1}{s}\end{aligned}$$

Note:
Using such formula is tedious. Most of the engineer use a Table with a partial fraction method instead.

Ramp function
For the ramp signal $f(t) = bt\mathbb{1}(t)$, we have

$$F(s) = \int_0^\infty bte^{-st}dt = \left[-\frac{bte^{-st}}{s} - \frac{be^{-st}}{s^2}\right]_0^\infty = \frac{b}{s^2},$$

where we used the technique of integration by parts,

$$\int udv = uv - \int vdu$$

with $u = bt$ and $dv = e^{-st}dt$.

Impulse function
For the impulse function $\delta(t)$, we have

$$F(s) = \int_{0^-}^\infty \delta(t)e^{-st}dt = e^{-s0} = 1,$$

by sampling property of the impulse function.

Sinusoid function
Find the laplace transform of the sinusoid function.

$$\mathcal{L}\left[\sin\omega t\right] = \int_0^\infty (\sin\omega t)e^{-st}dt$$

Using the relation

$$\begin{aligned}\sin\omega t &= \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}dt \\ &= \frac{1}{2j}\int_0^\infty \left(e^{(j\omega - s)t} - e^{-(j\omega + s)t}\right)dt \\ &= \frac{\omega}{s^2 + \omega^2}\end{aligned}$$

Laplace Transform Review

Properties of Laplace Transform

- **Superposition:** Since the Laplace transform is linear, then

$$\mathcal{L}\left[\alpha f_1(t) + \beta f_2(t)\right] = \alpha F_1(s) + \beta F_2(s)$$

  and

$$\mathcal{L}\left[\alpha f(t)\right] = \alpha F(s)$$

- **Time Delay:** Suppose a function $f_1(t) = f(t-T)$, which is delayed by $T > 0$, then

$$F_1(s) = \int_0^\infty f(t-T)e^{-st}dt = e^{-sT}F(s)$$

- **Time Scaling:** Suppose a function $f_1(t) = f(at)$, where t is scaled bya factor a, we have

$$F_1(s) = \int_0^\infty f(at)e^{-st}dt = \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

- **Shit in Frequency:** Suppose a function $f_1(t) = e^{at}f(t)$, the Laplace transform is

$$F_1(s) = \int_0^{\infty} e^{-at} f(t) e^{-st} dt = F(s+a)$$

- **Differentiation:**

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_{0^-}^{\infty} \left(\frac{df}{dt}\right) e^{-st} dt = -f(0^-) + sF(s)$$

$$\mathcal{L}\left[\ddot{f}(t)\right] = s^2 F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\mathcal{L}\left[f^{(m)}(t)\right] = s^m F(s) - s^{(m-1)} f(0^-) - \cdots - f^{(m-1)}(0^-)$$

- **Integration:**

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{1}{s} F(s)$$

- **Convolution:** The convolution in the time domain corresponds to multiplication in the S-domain. Assume that $\mathcal{L}[f_1(t)] = F_1(s)$ and $\mathcal{L}[f_2(t)] = F_2(s)$. Then

$$\mathcal{L}[f_1(t) * f_2(t)] = \int_0^\infty f_1(t) * f_2(t) e^{-st} dt = F_1(s) F_2(s)$$

$$\mathcal{L}^{-1}[F_1(s) F_2(s)] = f_1(t) * f_2(t)$$

- **Time product:** The multiplication in the time domain corresponds to convolution in the S-domain:

$$\mathcal{L}[f_1(t) f_2(t)] = \frac{1}{2\pi j} F_1(s) * F_2(s)$$

- **Multiplication by Time:** Multiplication by time corresponds to differentiation in the S-domain:

$$\mathcal{L}[t f(t)] = -\frac{d}{ds} F(s)$$

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a *partial-fraction expansion*.

Using this method,

$$F(s) = \frac{N(s)}{D(s)},$$

where the order of $N(s)$ must be less than the order of $D(s)$. If it is not, we can use the long division to find the remainder whose numerator is of order less than its denominator. For example, if

$$F(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} = s + 1 + \frac{2}{s^2 + s + 5}$$

Using the Laplace transform table, we obtain

$$f(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathcal{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$ the last term could be solve by partial-fraction.

Example:

$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}, \qquad \text{Find } y(t).$$

By partial fraction expansion:

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+3}$$

Using the cover-up method, we get

$$C_1 = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = \frac{8}{3}$$

$$C_2 = \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -\frac{3}{2}$$

$$C_3 = \left.\frac{(s+2)(s+4)}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

From the table the inverse Laplace transform of $Y(S)$ is

$$y(t) = \frac{8}{3}\mathbb{1}(t) - \frac{3}{2}e^{-t}\mathbb{1}(t) - \frac{1}{6}e^{-3t}\mathbb{1}(t).$$

There are two ways to solve the inverse Laplace transform using `Matlab`.

Matlab I

```
% using residue and table
num = conv([1 2],[1 4]);
% numerator

den = conv([1 1 0],[1 3]);
% denominator

[r,p,k] = residue(num,den);
% compute the residues
% r = [-0.1667, -1.5000, 2.6667]
% p = [-3, -1, 0]
```

Matlab II: symbolic

```
% using symbolic toolbox
syms s     % define a variable s

f = ilaplace((s+2)*(s+4)/...
     ((s*(s+1)*(s+3))));
pretty(f)
% result
  8   exp(-3 t)   3 exp(-t)
  - - --------- - ---------
  3       6           2
```

Scilab I

```
// define polynomial e.g.
// 1 + 2s^2 => [1 0 2]
// in matlab we use
// 1+2s^2 => [2 0 1]
num = convol([2 1],[4 1]);
// numerator
den = convol([0 1 1],[3 1]);
// denominator
Ns = poly(num,'s',"c");
Ds = poly(den,'s',"c");

[r] = residu(Ns/Ds);
// compute the residues
// r = [-0.1667, -1.5000, 2.6667]
// note 4.433D-17 is zero
```

Scilab I: result

```
 -->r
r  =  r(1)

      2.6666667
    -------------
    5.697D-17 + s

       r(2)

   - 1.5
    -----
    1 + s

       r(3)

  - 0.1666667
    ---------
      3 + s
```

Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 32y(t) = 32\mathbb{1}(t)$$

Taking the Laplace transform of the differential equation and set all initial conditions to zero is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}.$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)} = \frac{1}{s} + \frac{-2}{(s+4)} + \frac{1}{(s+8)}$$

From Laplace transform table, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence

$$y(t) = \left(1 - 2e^{-4t} + e^{-8t}\right)\mathbb{1}(t)$$

Consider $F(s) = \dfrac{2}{(s+1)(s+2)^2}$. In this case the partial fraction expansion is

$$F(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

Using a cover up method, $A = 2$ and $C = -2$. Letting $s = 0$ and substituting into above equation, then

$$F(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

From the Laplace transform table,

$$y(t) = \left(2e^{-1t} - 2te^{-2t} - 2e^{-2t}\right) \mathbb{1}(t)$$

For the higher degree of the repeated root, we could use "short-cut" method to solver for the solutions.

In this case, the most convenient way is using the frequency shift property. For example

$$F(s) = \frac{3}{s(s^2+2s+5)}.$$

By partial fraction expansion, we have

$$F(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

Using cover up and short-cut method, $A = 3/5$, $B = -3/5$, and $C = -6/5$. Then

$$\begin{aligned} F(s) &= \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5} \\ &= \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)2}{(s+1)^2+2^2} \end{aligned}$$

From the Laplace transform table, we obtain

$$f(t) = \left(\frac{3}{5} - \frac{3}{5}\left(e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\right)\right)\mathbb{1}(t) = \left(0.6 - 0.671e^{-t}\cos(2t - 26.57^\circ\right)\mathbb{1}(t)$$

Transfer function

Transfer Fucntion

A **Transfer Function** is the ratio of the output of a system to the input of a system, in the Laplace domain considering its initial conditions and equilibrium point to be zero.

A monic nth-order, linear, time-invariant differential equation,

$$\frac{d^n}{dt^n}y(t)+a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t)+\cdots+a_1\frac{d}{dt}y(t)+a_0y(t)= \\ b_m\frac{d^m}{dt^m}u(t)+b_{m-1}\frac{d^{m-1}}{dt^{m-1}}u(t)+\cdots+b_1\frac{d}{dt}u(t)+b_0u(t), \qquad n\geq m$$

Taking the Laplace transform to the both sides and set all initial conditions to be zero, the system becomes

$$\left(s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0\right)Y(s)=\left(b_ms^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0\right)U(s)$$

$$\frac{Y(s)}{U(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=G(s)$$

Transfer function

Example

Find the transfer function of the system represented by

$$\frac{d}{dt}y(t) + 2y(t) = u(t)$$

Taking the Laplace transform of both sides, and set all initial conditions to be zero, we have

$$sY(s) + 2Y(s) = U(s)$$

The tranfer function, $G(s)$ is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+2}$$

Matlab Coode I

```
num = [1];
den = [1 2];

sys = tf(num,den)
```

Matlab Code II

```
s = tf('s');

sys = 1/(s+2);
```

Transfer function

Example

Scilab Coode I

```
num = poly([1,0],'s',"c");
den = poly([2 1],'s',"c");

sys = syslin('c',num,den)
```

Scilab Code II

```
num = 1;
den = %s + 2;
// %s is a variable s
sys1 = syslin('c',num,den);
```

Scilab Code III

```
 s = poly(0,'s');

sys1 = 1/(s+2)
```

Find the response, $y(t)$ to an $\mathbb{1}(t)$ input of a system $G(s) = 1/(s+2)$.
The Laplace transform of $\mathbb{1}(t)$ is $1/s$. Then

$$Y(s) = G(s)U(s) = \frac{1}{(s+2)}\frac{1}{s}$$

Using the partial fraction expansion, we get

$$Y(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

Finally, taking the inverse Laplace transform of each term yields

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$$

We can use a Matlab command `step(G)` to get the response of the system to the unit-step input, which gives the same result as directly time domain calculating with Matlab.

Transfer function

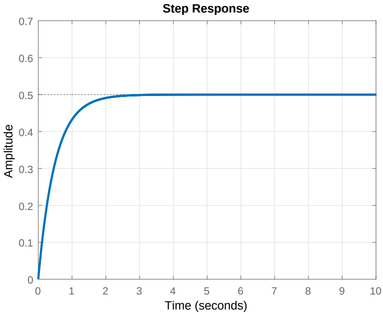
System Response from the Transfer Function

Matlab Code I

```
num = [1];
den = conv([1 0],[1 2]);
G = tf(num,den);
t = 0:0.01:10;

step(G,t);
```

Matlab Code II

```
syms s
G = 1/(s*(s+2));

y = ilaplace(G);
t = 0:0.01:10;
plot(t,eval(y));
```

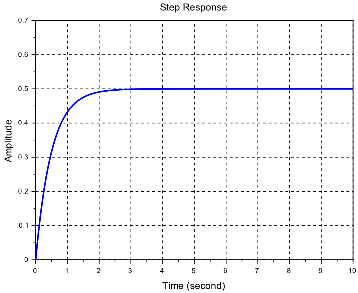


Scilab Code I

```
num = 1;
den = %s*(%s +2);
G = syslin('c',num,den);
t = 0:0.01:10;
y = csim('step',t,G);
plot(t,y)
```

Scilab Code II

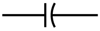
```
s = poly(0,'s');
G = 1/(s*(s+2));
t = 0:0.01:10;

y = csim('step',t,G);
plot(t,y)
```



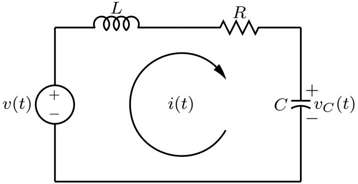
Transfer function

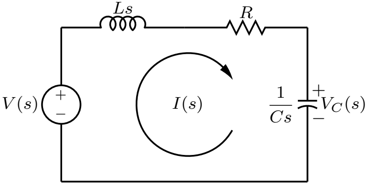
Electrical Network

In feedback control system design, we use electrical networks to build analog controllers, analog filters, etc. These are RLC circuits and operational amplifier circuits.

Component	Voltage-current	Current-voltage	Impedance $Z(s) = V(s)/I(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{d}{dt}v(t)$	$\frac{1}{Cs}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	R
 Inductor	$v(t) = L\frac{d}{dt}i(t)$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	Ls

Note: All initial conditions are zero.





The Laplace transform of the mesh voltage with all zero conditions, is

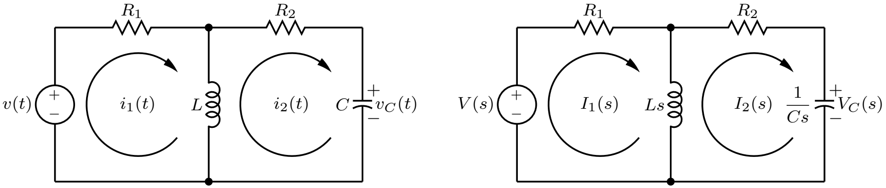
$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = V(s)$$

$$\left(Ls + R + \frac{1}{Cs}\right) CsV_C(s) = V(s)$$

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Transfer function

Electrical Network: RLC circuit



$$\begin{bmatrix} R_1 + Ls & -Ls \\ -Ls & R_2 + Ls + \frac{1}{Cs} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \frac{1}{(R_1 + Ls)(R_2 + Ls + \frac{1}{Cs}) - L^2 s^2} \begin{bmatrix} R_2 + Ls + \frac{1}{Cs} & Ls \\ Ls & R_1 + Ls \end{bmatrix} \begin{bmatrix} V(s) \\ 0 \end{bmatrix}$$

$$I_2(s) = \frac{LsV(s)}{(R_1 + Ls)(R_2 + Ls + \frac{1}{Cs}) - L^2 s^2}$$

$$\frac{I_2(s)}{V(s)} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1 R_2 C + L)s + R_1}$$

Transfer function

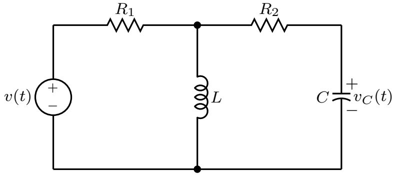
Electrical Network: RLC circuit

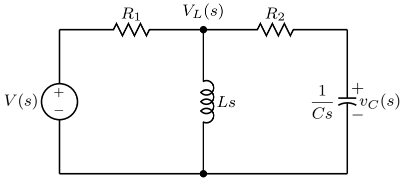
Matlab code

```
syms R1 R2 V L C s

A = [ R1 + L*s, -L*s;  -L*s, R2+L*s+1/(C*s)];
b = [V ; 0];

I = inv(A)*b;
I2 = I(2,1);

% find the transfer function I2/V
sys = I2/V
pretty(sys)

                        2
                   C L s
--------------------------------------------
                   2            2
R1 + L s + C L R1 s  + C L R2 s  + C R1 R2 s
```

Transfer function

Electrical Network: RLC circuit





$$\frac{V_L(s)-V(s)}{R_1}+\frac{V_L(s)}{Ls}+\frac{V_L(s)-V_C(s)}{R_2}=0$$

$$\frac{V_C(s)-V_L(s)}{R_2}+CsV_C(s)=0$$

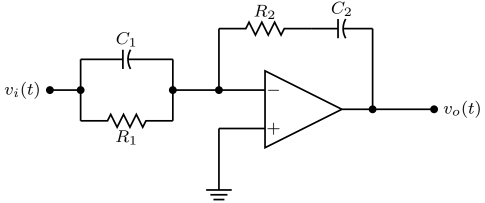
$$V_L(s)=(R_2Cs+1)V_C(s)$$

Substituting $V_L(s)$ to the first equation, we have

$$R_2Ls(R_2Cs+1)V_C(s)+R_1R_2(R_2Cs+1)V_C(s)+R_1R_2LCs^2V_C(s)=R_2LsV(s)$$

$$\frac{V_C(s)}{V(s)}=\frac{Ls}{(R_1+R_2)LCs^2+(L+R_1R_2C)s+R_1}$$

Instead of consider R, L and C separately, each composition RL, RC, LC, and RLC could be considered in terms of impedance as follow.



$$Z_1(s) = C_1||R_1 = \frac{R_1}{R_1C_1s+1}$$

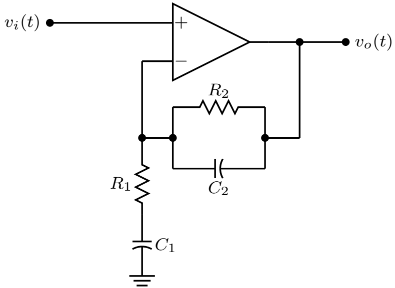
$$Z_2(s) = R_2 + \frac{1}{Cs}$$

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2 + \frac{1}{Cs}}{\frac{R_1}{R_1C_1s+1}} = -\frac{(R_1C_1s+1)(R_2C_2s+1)}{R_1C_2s}$$

This circuit is called a PID controller.

Transfer function

Electrical Network: Noninverting Amplifier



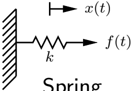
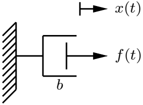
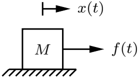
$$Z_1(s) = R_1 + \frac{1}{C_1 s}$$

$$Z_2(s) = R_2 || C_2 = \frac{R_2}{R_2 C_2 s + 1}$$

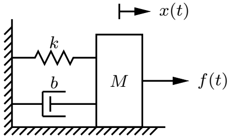
$$\frac{V_o(s)}{V_i(s)} = 1 + \frac{Z_2(s)}{Z_1(s)} = 1 + \frac{\dfrac{R_2}{R_2 C_2 s + 1}}{\dfrac{R_1 C_1 s + 1}{C_1 s}} = 1 + \frac{R_2 C_1 s}{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}$$

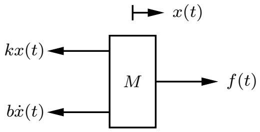
$$= \frac{R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_1 + R_2 C_2) s + 1}{(R_1 R_2 C_1 C_2 s^2 + (R_1 C_1 + R_2 C_2) s + 1}$$

Transfer function
Translational Mechanical System

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
 Spring	$f(t) = k\int_0^t v(\tau)d\tau$	$f(t) = kx(t)$	k
 Viscous damper	$f(t) = bv(t)$	$f(t) = b\dot{x}(t)$	bs
 Mass	$f(t) = M\dot{v}(t)$	$f(t) = M\ddot{x}$	Ms^2

Assuming that there are no friction between a mass and ground.





From the Newton's law $\Sigma F = ma$, we have

$$M\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t)$$

Taking the Laplace transform of above equation and setting all initial conditions to be zero, we obtain

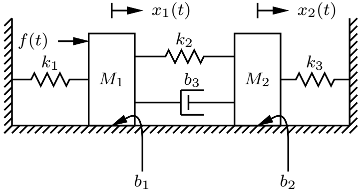
$$Ms^2X(s) + bsX(s) + kX(s) = F(s)$$

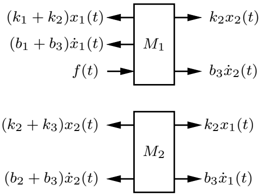
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + k}$$

Transfer function

Translational Mechanical System: Example

The point of motion in a system can still move if all other points of motion are held still. The name for the number of the linearly independent motions i sthe number of the degrees of freedom.





$$M_1\ddot{x}_1(t)+(b_1+b_3)\dot{x}_1(t)-b_3\dot{x}_2(t)+(k_1+k_2)x_1(t)-k_2x_2=f(t)$$
$$M_2\ddot{x}_2(t)+(b_2+b_3)\dot{x}_2(t)-b_3\dot{x}_1(t)+(k_2+k_3)x_2(t)-k_2x_1=0$$

Taking the Laplace transform of both equations, we get

$$\left(M_1 s^2 + (b_1 + b_3)s + (k_1 + k_2)\right) X_1(s) - (b_3 s + k_2)X_2(s) = F(s)$$
$$-(b_3 s + k_2)X_1(s) + \left(M_2 s^2 + (b_2 + b_3)s + (k_2 + k_3)\right) X_2(s) = 0$$

Rearranging the equations into matrix form:

$$\begin{bmatrix} M_1 s^2 + (b_1 + b_3)s + (k_1 + k_2) & -(b_3 s + k_2) \\ -(b_3 s + k_2) & M_2 s^2 + (b_2 + b_3)s + (k_2 + k_3) \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

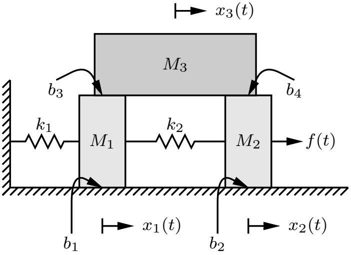
$$\frac{X_2(s)}{F(s)} = \frac{b_3 s + k_2}{\Delta}$$

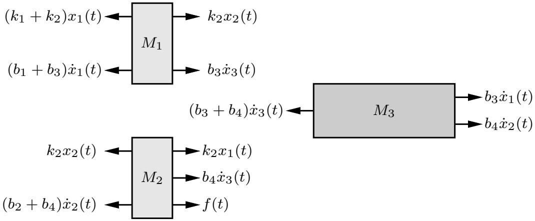
where

$$\Delta = \begin{vmatrix} M_1 s^2 + (b_1 + b_3)s + (k_1 + k_2) & -(b_3 s + k_2) \\ -(b_3 s + k_2) & M_2 s^2 + (b_2 + b_3)s + (k_2 + k_3) \end{vmatrix}$$

Transfer function

Translational Mechanical System: Example





The equations of motion are

$$\begin{aligned} M_1\ddot{x}_1(t)+(b_1+b_3)\dot{x}_1(t)+(k_1+k_2)x_1(t)-k_2x_2(t)-b_3\dot{x}_3(t)&=0\\ M_2\ddot{x}_2(t)+(b_2+b_4)\dot{x}_2(t)+k_2x_2(t)-k_2x_1(t)-b_4\dot{x}_3(t)&=f(t)\\ M_3\ddot{x}_3(t)+(b_3+b_4)\dot{x}_3(t)-b_3\dot{x}_1(t)-b_4\dot{x}_2(t)&=0 \end{aligned}$$

Transfer function

Translational Mechanical System: Example

Taking the Laplace transform to all equations, we have

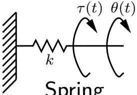
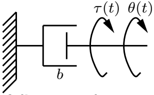
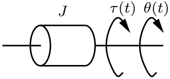
$$\begin{aligned}\left(M_1 s^2 + (b_1+b_3)s + (k_1+k_2)\right) X_1(s) - k_2 X_2(s) - b_3 s X_3(s) &= 0\\ -k_2 X_1(s) + \left(M_2 s^2 + (b_2+b_4)s + k_2\right) X_2(s) - b_4 s X_3(s) &= F(s)\\ -b_3 s X_1(s) - b_4 s X_2(s) + \left(M_3 s^2 + (b_3+b_4)s\right) X_3(s) &= 0\end{aligned}$$

and in matrix from

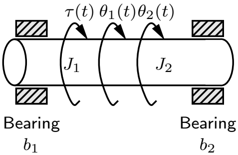
$$\begin{bmatrix} M_1 s^2 + (b_1+b_3)s + (k_1+k_2) & -k_2 & -b_3 s\\ -k_2 & M_2 s^2 + (b_2+b_4)s + k_2 & -b_4 s\\ -b_3 s & -b_4 s & M_3 s^2 + (b_3+b_4)s \end{bmatrix} \begin{bmatrix} X_1(s)\\ X_2(s)\\ X_3(s)\end{bmatrix} = \begin{bmatrix} 0\\ F(s)\\ 0\end{bmatrix}$$

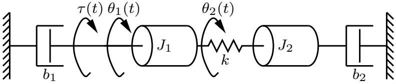
Note: the matrix is symmetry.

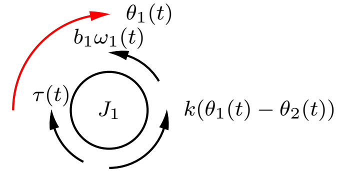
Transfer function
Rotational Mechanical System

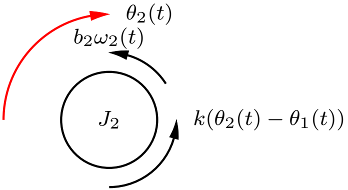
Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = \hat{\tau}(s)/\Theta(s)$
 Spring	$\tau(t) = k\int_0^{t_1} \omega(t)dt$	$\tau(t) = k\theta(t)$	k
 Viscous damper	$\tau(t) = b\omega(t)$	$\tau(t) = b\dot{\theta}(t)$	bs
 Inertia	$\tau(t) = J\dot{\omega}(t)$	$\tau(t) = J\ddot{\theta}$	Js^2

Transfer function
Rotational Mechanical System: Example









The equations of motion are

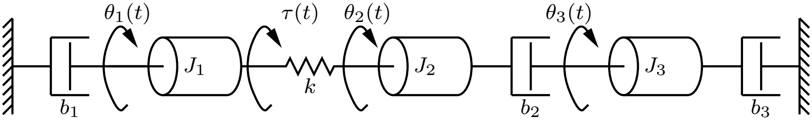
$$\begin{aligned} \tau(t) - b_1\dot{\theta}_1(t) - k(\theta_1(t)-\theta_2(t)) &= J_1\ddot{\theta}_1 \\ -k(\theta_2(t)-\theta_1(t)) - b_2\dot{\theta}_2(t) &= J_2\ddot{\theta}_2 \end{aligned}$$

Taking the Laplace transform, we have

$$\begin{aligned} \left(J_1 s^2 + b_1 s + k\right)\Theta_1(s) - k\Theta_2(s) &= \hat{\tau}(s) \\ \left(J_2 s^2 + b_2 s + k\right)\Theta_2(s) - k\Theta_1(s) &= 0 \end{aligned}$$

Transfer function

Rotational Mechanical System: Example



The equations of motion are

$$
\begin{aligned}
\tau(t) - b_1\dot{\theta}_1(t) - k\left(\theta_1(t) - \theta_2(t)\right) &= J_1\ddot{\theta}_1 \\
-k\left(\theta_2(t) - \theta_1(t)\right) - b_2\left(\dot{\theta}_2 - \dot{\theta}_3\right) &= J_2\ddot{\theta}_2 \\
-b_2\left(\dot{\theta}_3 - \dot{\theta}_2\right) - b_3\dot{\theta}_3 &= J_3\ddot{\theta}_3
\end{aligned}
$$

Taking the Laplace transform, we have

$$
\begin{aligned}
\left(J_1 s^2 + b_1 s + k\right)\Theta_1(s) - k\Theta_2(s) &= \hat{\tau}(s) \\
-k\Theta_1(s) + \left(J_2 s^2 + b_2 s + k\right)\Theta_2(s) - b_2 s\Theta_3(s) &= 0 \\
-b_2 s\Theta_2(s) + \left(J_3 s^2 + (b_2 + b_3)s\right)\Theta_3(s) &= 0
\end{aligned}
$$

Transfer function

Rotational Mechanical System: Example

In matrix from

$$\begin{bmatrix} \left(J_1 s^2 + b_1 s + k\right) & -k & 0 \\ -k & \left(J_2 s^2 + b_2 s + k\right) & -b_2 s \\ 0 & -b_2 s & \left(J_3 s^2 + (b_2 + b_3) s\right) \end{bmatrix} \begin{bmatrix} \Theta_1(s) \\ \Theta_2(s) \\ \Theta_3(s) \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\tau}(s) \\ 0 \\ 0 \end{bmatrix}$$

Reference

1. Norman S. Nise, "*Control Systems Engineering*,6^{th} edition, Wiley, 2011
2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002