INC 342 Feedback Control Systems: Lecture 12 Lead-Lag Compensator

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Phase-Lead Compensator

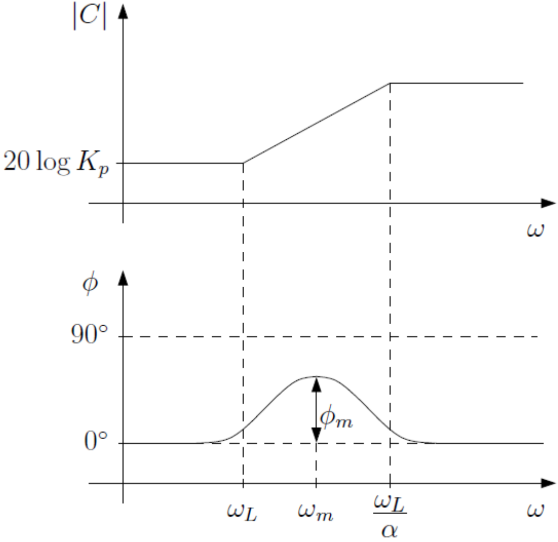
- If the phase of the plant transfer function $G(s)$ at the crossover frequency ω_c does not provide a sufficient phase margin, the controller transfer function $C(s)$ can be used to compensate for this phase lag.
- An additional zero with corner frequency below the crossover frequency increases the phase angle by 90°
- In fact, introducing an additional zero is equivalent to introducing derivative feedback.
- The transfer function of a *phase-lead compensator* is

$$C(s) = K_p \frac{1 + T_L s}{1 + \alpha T_L s},$$

  where $\alpha < 1$.
- The corner frequency $\omega_L = 1/T_L$, this can be written as

$$C(j\omega) = K_p \frac{1 + j\frac{\omega}{\omega_L}}{1 + j\frac{\omega}{\omega_L/\alpha}}$$

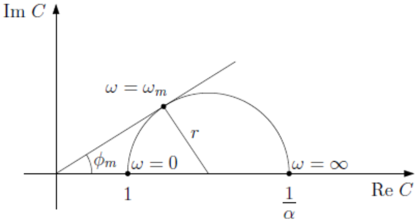
Phase-Lead Compensator



Phase-Lead Compensator

- From the previous picture, the phase maximum is located at the frequency ω_m.
- the phase angle at ω_m depends on the "distance" between the pole and the zero of the compensator.
- If the distance is large, say three decades (obtained by taking $\alpha = 0.001$), a phase increase of almost $90°$, however at the expense of a 60 dB gain increase at high frequencies.
- The compensator design involves a trade-off between phase increase an low noise sensitivity, and the distance between ω_L and ω_L/α should just be large enough to obtain the desired phase margin, without unnecessarily increasing the high frequency gain.
- When designing a phase-lead compensator, there are three design parameters to be chosen: the proportional gain K_p, the corner frequency ω_L and the parameter α.
- An exact relationship between the maximum phase angle ϕ_m and the parameter α can be derived from the Nyquist plot of the compensator.

Phase-Lead Compensator



- Assume $K_p = 1$, the Nyquist plot for the phase-lead compensator is a semicircle with its position determined by

$$\omega = 0 \;\Rightarrow\; C = 1$$
$$\omega = \infty \;\Rightarrow\; C = \frac{1}{\alpha} > 1$$

and the fact that the phase for frequencies $0 < \omega < \infty$ is positive.

Phase-Lead Compensator

- The point where the phase attains its maximum ϕ_m is marks. The radius of the semicircle is

$$r = \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)$$

- we have

$$\sin\phi_m = \frac{r}{1+r} = \frac{\alpha r}{\alpha + \alpha r} \quad \text{or} \quad \sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

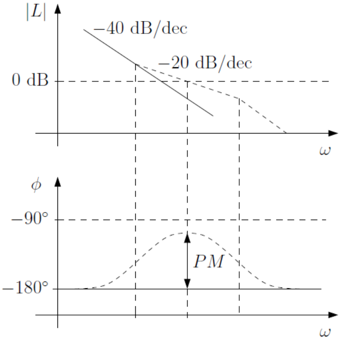
- Solving for α yields

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m}$$

Phase-Lead Compensator

- When the parameter α has been determined to achieve the desired phase lead, the corner frequency ω_L must be chosen such that the phase maximum is located at the crossover frequency, i.e. ω_m should be equal to the crossover frequency.
- The uncompensated loop gain has a slope of -40 dB/dec, and the lead compensator is designed to increase the phase at the crossover frequency.

Phase-Lead Compensator



Taking the log scale of the frequency axis into account, we have

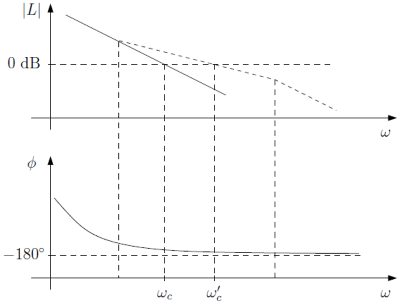
$$\log \omega_m = \frac{1}{2}\left(\log \omega_L + \log \frac{\omega_L}{\alpha}\right)$$

$$\omega_L = \sqrt{\alpha}\omega_m$$

Phase-Lead Compensator

The design procedure can be summarized as follows:

- Step 1 Choose K_p, find ω_c
- Step 2 Take $\omega_m = \omega'_c (\approx 1.5 \ldots 2 \cdot \omega_c)$
- Step 3 Determine ϕ_m from the phase angle at ω'_c
- Step 4 Compute α
- Step 5 Compute ω_L



Phase-Lag Compensator

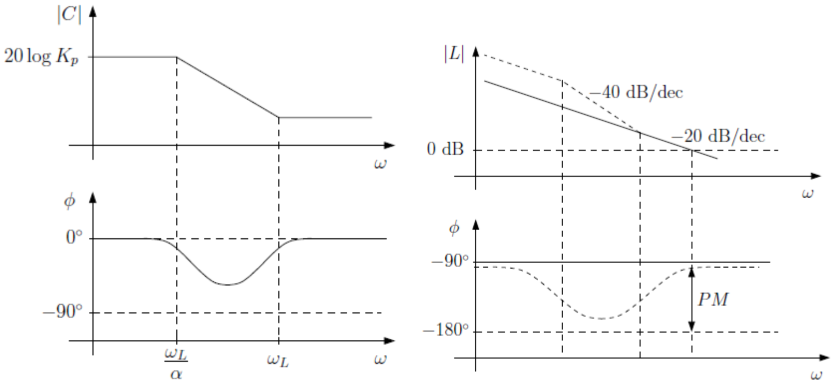
- The phase-lag compensator is

$$C(s) = K_p \frac{1 + T_L s}{1 + \alpha T_L s}$$

where $\alpha > 1$. The only difference to a phase-lead compensator is that α is greater than 1.

- As a result, the corner frequency associated with the pole is lower than that of the zero, and the effect on gain and phase is reversed.
- Because of the decrease in phase such a compensator is called a **phase-lag compensator**.
- The desired effect of a phase-lag compensator is the change in the loop gain, whereas the change in phase is an undesired side effect.
- This is in contrast to phase-lead compensation, where the phase change is the desired effect and the gain change is an undesired side-effect.
- When designing a phase-lag compensator, care must thus be taken to locate the corner frequencies well below (or well above) the crossover frequency such that the phase lag does not reduce the phase margin.

Phase-Lag Compensator

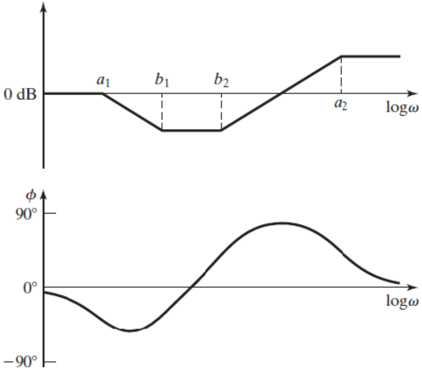


- Frequency response of a phase-lag compensator (left).
- Increasing the low frequency the low frequency gain by phase-lag compensator (right).

Phase-Lead-Lag Compensator

A lead-lag compensator can take the following general form:

$$C(s) = K_p \left(\frac{1 + \frac{s}{b_1}}{1 + \frac{s}{a_1}} \right) \left(\frac{1 + \frac{s}{b_2}}{1 + \frac{s}{a_2}} \right), \quad a_1 < b_1 < b_2 < a_2.$$



Assume $K_p = 1$

Phase-Lead-Lag Compensator

The following procedure can be use to design a lead-lag compensator.

Step 1 Find K_p so that the DC gain requirements of the open-loop system $L(s) = K_p G(s)$ are satisfied. For example, K_p has to be chosen to satisfy steady-state error requirements on tracking, disturbance rejection, and so on.

Step 2 Determine the desired crossover frequency ω_c and the phase margin (PM) ϕ_{desired}

Step 3 Plot the Bode diagram of $K_p G(s)$ and calculate the phase ϕ_m needed at ω_c in order to achieve that desired PM:

$$\phi_m = \phi_{\text{desired}} - \angle K_p G(j\omega_c) - 180^\circ + 5^\circ$$

(The reason for adding 5° at ω_c when the controller parameters are appropriately hosen as below.) The phase can be calculated precisely by using a command

`[mag, phase] = bode(K_P*G, wc)`.

Phase-Lead-Lag Compensator

Step 4 Choose a_2 and b_2 such that

$$\frac{a_2}{b_2} = \frac{1 + \sin \phi_m}{1 - \sin \phi_m}, \quad \omega_c = \sqrt{b_2 a_2}$$

Let

$$C_{\text{lead}}(s) = K_p \frac{1 + \frac{s}{b_2}}{1 + \frac{s}{a_2}}$$

Step 5 Choose

$$b_1 \approx 0.1\omega_c, \quad a_1 = \frac{b_1}{|C_{\text{lead}}(j\omega)\, G(j\omega_c)|}.$$

The magnitude can be calculated precisely by using a command

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[mag, phase] = bode(Clead*G, wc)
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Step 6 Plot the Bode diagram of $C(s)\,G(s)$ and check the design specifications.

Reference

1. Norman S. Nise, "*Control Systems Engineering*,6^{th} edition, Wiley, 2011
2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002
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4. Li Qiu and Kemin Zhou, "*Introduction to Feedback Control*", Prentice Hall, 2010