INC 342 Feedback Control Systems: Lecture 11 Frequency Domain Analysis

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Review of Bode plot

- The most widely used graphical representation of a frequency response is the Bode diagram, developed in the Bell laboratories in the 1930s by W. H. Bode.
- Magnitude and phase are plotted versus frequency in two separate plots, where a log scale is used for magnitude and frequency and a linear scale for the phase.
- The log scale is useful because the transfer function is composed of pole and zero factors which can be added graphically.
- For example

$$G(j\omega) = \frac{g_1(j\omega)g_2(j\omega)}{g_3(j\omega)}$$

Using the notation

$$g_i = |g_i|e^{j\varphi_i}$$

This can be written as

$$G = |G|e^{j\phi} = \frac{|g_1||g_2|}{|g_3|}e^{j(\phi_1+\phi_2-\phi_3)}$$

Review of Bode plot

- Thus

$$|G| = \frac{|g_1||g_2|}{|g_3|}, \; \phi = \varphi_1 + \varphi_2 - \varphi_3$$

- the phase of G is the sum of the phase angles of the factors, and on a log scale we also have

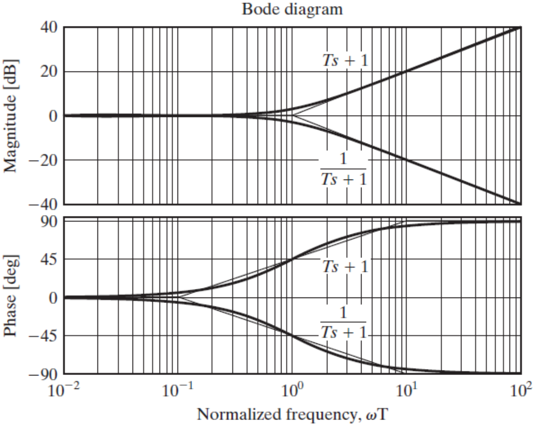
$$\log|G| = \log|g_1| + \log|g_2| - \log|g_3|$$

It is standard to measure the logarithmic gain $\log|G|$ in dB, the definition is

$$|G|_{\text{dB}} = 20\log|G|$$

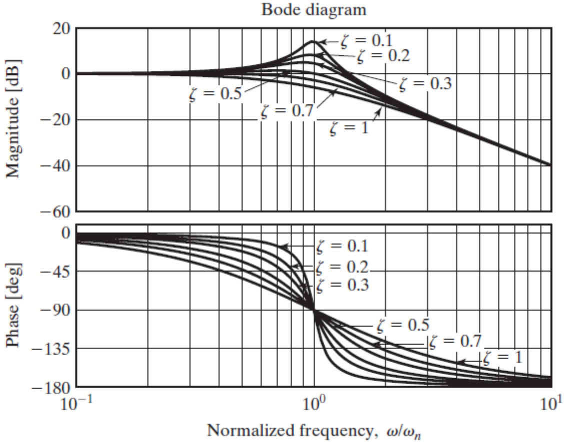
Review of Bode plot

First order term



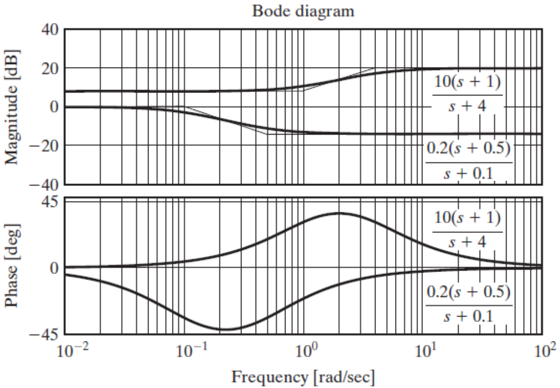
Review of Bode plot

Second order term



Review of Bode plot

Example



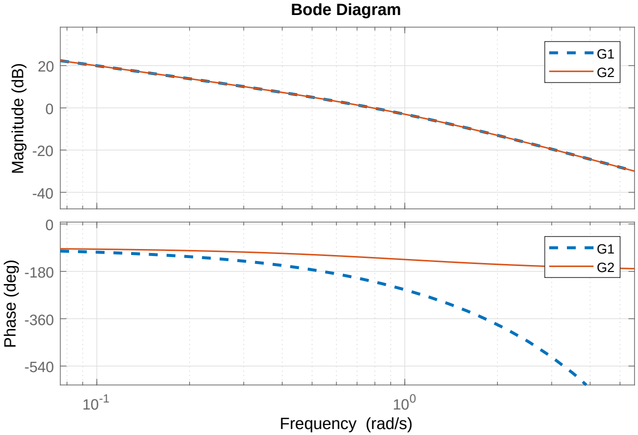
Review of Bode plot

Consider

$$G(s) = \frac{10(s+1)}{s+4}$$

```
s = tf('s');
G = 10*(s+1)/(s+4);
bode(G);
% or
w =  logspace(-2,2,200)
% set the frequency range from  10^-2 to 10^2 with 200 sample points
bode(G,w)
```

Review of Bode plot

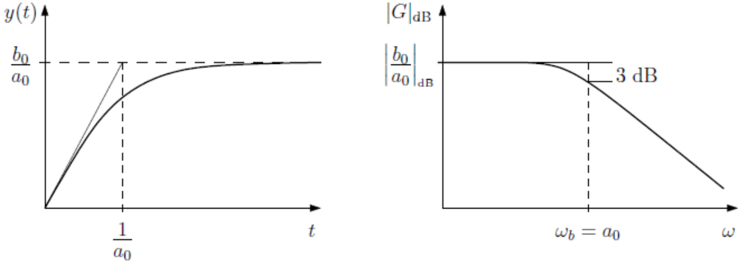


Delay time reduces the phase margin of the system.

Review of Bode plot

Bandwidth consider a first-order system

$$G(s) = \frac{b_0}{s + a_0} = \frac{b_0/a_0}{\frac{1}{a_0}s + 1} = \frac{K_c}{\tau s + 1}$$



Review of Bode plot

Bandwidth of the second order system

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

We can take

$$\omega_b \approx \omega_n$$

as a reasonable approximation of the bandwidth. We also have

$$\omega_b \approx \frac{1.7}{t_r},$$

where t_r is a rise-time of the second order system.

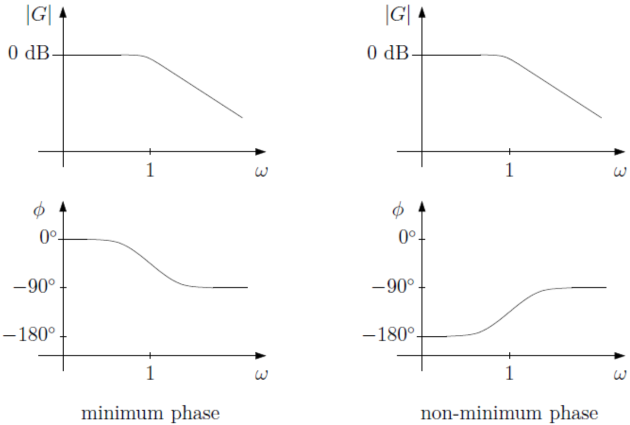
Review of Bode plot

minimum phase system A system with transfer function $G(s)$ is called *minimum phase* if it has no pole or zero on the right half plane

The definition implies that a minimum-phase system is stable; in addition there must be no zeros in the right half plane. Conversely, a system is *non-minimum phase* if it has right half plane zeros or poles or both.

$$G_1(s) = \frac{1}{s+1} \quad \text{stable and minimum-phase}$$
$$G_2(s) = \frac{1}{s-1} \quad \text{unstable and nonminimum-phase}$$

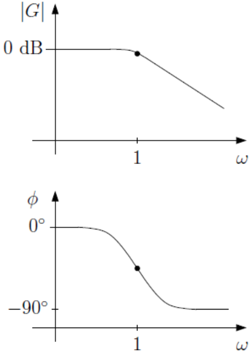
Review of Bode plot

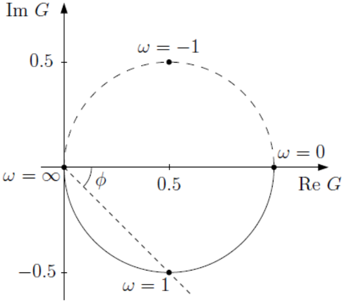


Nyquist Plot

Bode and Nyquist plot of

$$G(s) = \frac{1}{s+1}$$





Nyquist Plot

The idea of Nyquist plot, we consider the transfer function

$$G(s) = \frac{1}{s+1} \quad \text{or} \quad G(j\omega) = \frac{1}{j\omega + 1}$$

For $\omega = 0$ we have $G = 1$, and for $\omega = \infty$ we have $G = 0$. At the corner frequency $\omega = 1$

$$G(j\omega) = \frac{1}{1+j} = \frac{1}{2} - j\frac{1}{2}$$

Actually the magnitude and phase of $G(j\omega)$ can be read from the Bode plot.

- The value of the magnitude are 1 (or 0 dB) or 0 (or $-\infty$ dB) and the phase angles are 0° and -90°
- At the corner frequency, the magnitude is $1/\sqrt{2}$ (-3dB) and the phase is -45°.
- The complex values at the frequencies 0, 1 and ∞ are marked in the $G(s)$-plane.
- Doing this for sufficiently many frequencies reveals that the points are located on a semicircle centered at $1/2$; this semicircular plot is the Nyquist plot of $G(s)$ for positive frequencies.
- The Nyquist plot of negative frequencies is a mirror image of the positive frequencies.

Nyquist Plot

Consider a second order system

$$G(s) = \frac{1}{s(s+1)}$$

Evaluated on the imaginary axis

$$\begin{aligned} G(j\omega) &= \frac{1}{j\omega(j\omega+1)} = \frac{1}{-\omega^2 + j\omega} \\ &= \frac{-\omega^2 - j\omega}{\omega^4 + \omega^2} = -\frac{1}{\omega^2+1} - j\frac{1}{\omega(\omega^2+1)} \end{aligned}$$

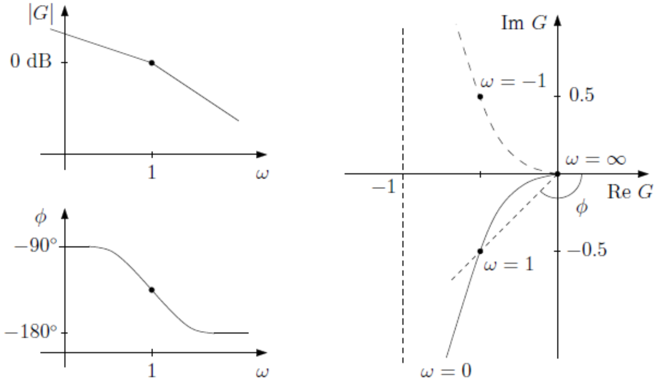
We have

$$\begin{aligned} \omega = 0 : &\qquad G = -1 - j\infty \\ \omega = 1 : &\qquad G = -\frac{1}{2} - j\frac{1}{2} \\ \omega = \infty : &\qquad G = 0 \end{aligned}$$

Nyquist Plot

Bode and Nyquist plot of

$$G(s) = \frac{1}{s(s+1)}$$



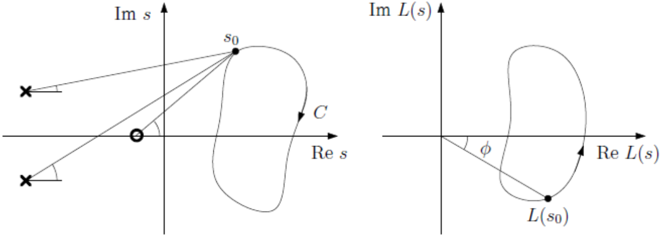
The Nyquist Stability Criterion

The Nyquist stability criterion is based on Cauchy's principle; the idea is to use the contour evaluation o fan open-loop transfer function to determine the presence of unstable closed-loop poles.

Assume we have a transfer function $L(s)$

$$L(s) = \frac{s - z}{(s - p_1)(s - p_2)}$$

The Nyquist Stability Criterion

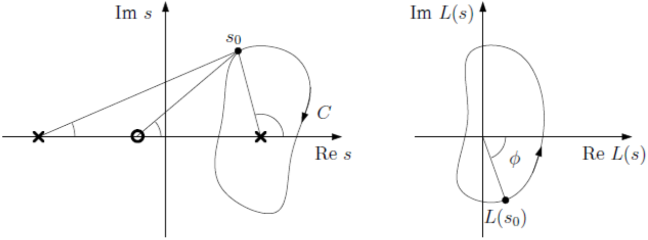


- In the s-plane, two poles and a zero are marked, together with a test point s_0 which is moved clockwise along a closed contour C.
- We are interested in the change of the phase angle of $L(s_0)$ when s_0 is move around the closed contour C.
- The phase angel of $L(s_0)$ is the sum of the zero angel and the negative pole angels indicated inteh plot.
- Full traverse, the change of the phase angle is zero.

The Nyquist Stability Criterion

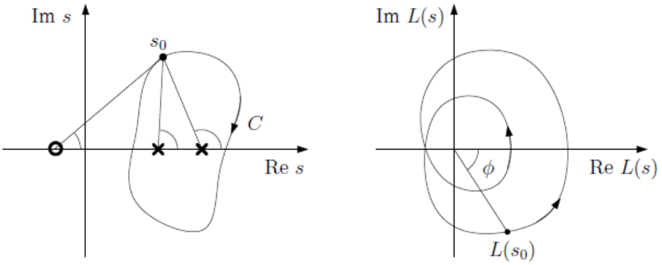
- The function $L(s)$ maps each point on C into the $L(s)$-plane, and the point $L(s_0)$ moves along a closed contour which is the mapping of C as s_0 move along C.
- ϕ denotes the phase angle of $L(s_0)$.
- The plot shows that the change of the phase angle of $L(s_0)$ after a full traverse is zero, i.e. $\Delta\phi = 0^\circ$.

The Nyquist Stability Criterion



- One of poles of $L(s)$ is enclosed by C.
- In this case the phase change of $L(s_0)$ after a full transverse is not zero: the angle of the enclosed pole undergoes a change of $360°$. If s_0 moves clockwise, then $L(s_0)$ move counterclockwise and we have $\Delta\phi = 360°$.
- This contour must encircle the origin of the $L(s)$-plane counterclockwise.
- If instead of a pole a zero is enclosed, $\Delta\phi = -360°$ and $L(s_0)$ encircles the origin clockwise.

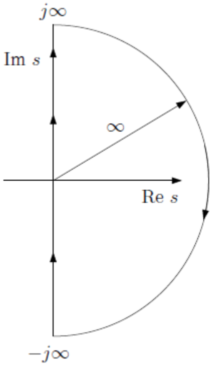
The Nyquist Stability Criterion



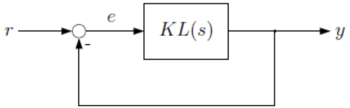
- Two poles are enclosed, and we have $\Delta\phi = 720^\circ$ after one traverse along C.
- Therefore the mapping of C must encircle the origin of the $L(s)$-plane two times counterclockwise.

The Nyquist Stability Criterion

Nyquist's idea was to use this fact to detect the presence of closed-loop poles in the right half plane. For this purpose, the contour C must be chosen such that it encloses the whole right half plane. Such a contour, known as the *Nyquist path*.



The Nyquist Stability Criterion

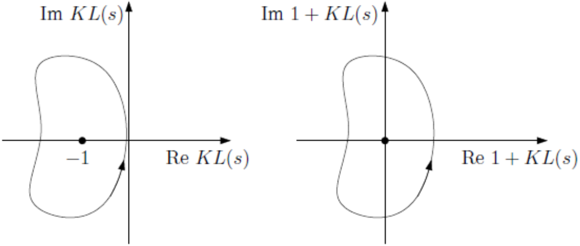


The closed-loop characteristic equation is

$$1 + KL(s) = 0$$

An unstable zero of $1 + KL(s)$ indicates an unstable closed-loop pole. For Nyquist plot, instead of evaluating $1 + KL(s)$ and checking encirclements of the origin, we can equivalently evaluate the open-loop transfer function $KL(s)$ and check encirclements of the point -1

The Nyquist Stability Criterion



- To evaluate $KL(s)$ along the Nyquist path means to evaluate along the $j\omega$-axis from $\omega = -\infty$ to $\omega = \infty$, and along the infinite arc.
- However, the transfer functions of physical systems are zero at infinite frequency, and the infinite arc in the s-plane is mapped into the origin of the $L(s)$-plane.
- Therefore, $KL(s)$ needs to be evaluated only from $-j\infty$ to $j\infty$.
- This yields precisely the Nyquist plot of $KL(s)$ for positive and negative frequencies.

The Nyquist Stability Criterion

- The number of encirclements of the point -1 is not only influenced by the *zeros* but also by *poles* of $1 + KL(s)$ in the right half plane.
- If the right half plane contains a zero or a pole of $1 + KL(s)$, the Nyquist plot of $KL(s)$ encircles the point -1 (clockwise or counterclockwise, respectively).
- To distinguish between poles and zeros of the function $1 + KL(s)$, we write $L(s) = b(s)/a(s)$ to obtain

$$1 + KL(s) = \frac{a(s) + Kb(s)}{a(s)}$$

- the *zeros* of the function $1 + KL(s)$ are the *closed-loop poles*, and
- the *poles* of the function $1 + KL(s)$ are the *open-loop poles* (the poles of $L(s)$)
- If the open loop transfer function is unstable, the number of unstable poles is known. So that can be taken into account when checking closed-loop stability.
- If we let Z denote the number of unstable closed-loop pole, and P the number of unstable open-loop poles, then the above considerations show that the Nyquist plot of $KL(s)$ encircles $N = Z - P$ times clockwise the point -1.

The Nyquist Stability Criterion

Nyquist Stability Test

1. Draw the Nyquist plot of $KL(s)$
2. Determine the number N of clockwise encirclements of the point -1 and the number P of unstable open-loop poles
3. The closed-loop system has $Z = N + P$ unstable poles.

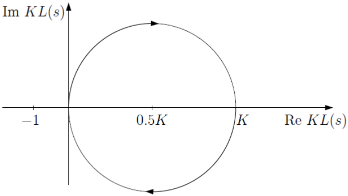
If the open loop transfer function is stable, then the closed-loop system is stable if the Nyquist polt does not encircle the point -1. If $KL(s)$ has one unstable pole, then closed-loop stability requires one counterclockwise encirclement of -1.

The Nyquist Stability Criterion: Example

Consider

$$KL(s) = \frac{K}{s+1}$$

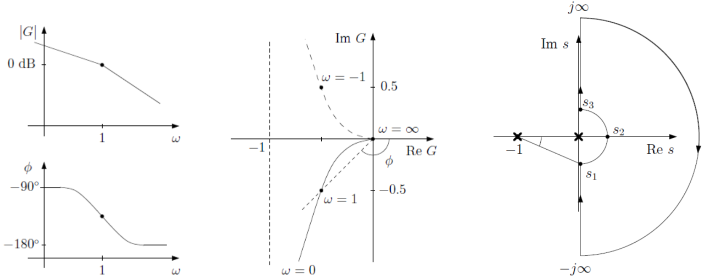
- The Nyquist plot is shown in Figure, it is a circle centered at $K/2$ with radius $K/2$
- Because $P = 0$ (no unstable poles of $L(s)$) and $N = 0$ (no encirclement of -1) for all $K > 0$
- The closed-loop system is stable for all positive K.



The Nyquist Stability Criterion: Example

Consider

$$KL(s) = \frac{K}{s(s+1)}$$



- The Nyquist plot is in fact closed by an arc at infinity.
- The Nyquist path has been modified to avoid the pole at the origin by making an infinitely small detour to the right.

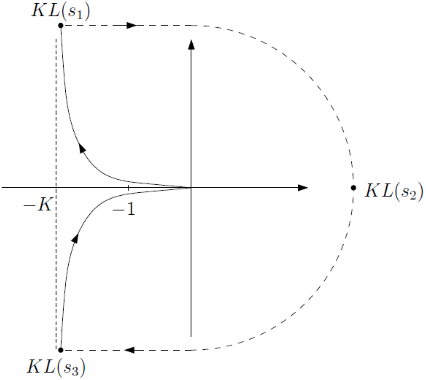
The Nyquist Stability Criterion: Example

- It is this small arc that is mapped into an arc at infinity, and to determine whether or not the Nyquist plot encircles the point -1.
- It is important to know whether this infinite arc encloses the right half plane or the left half plant.
- One the small arc, three test points $s_i, i = 1, 2, 3$ are marked.
- Because the radius of the semicircle around the origin is infinitely small, the magnitude of $KL(s_i)$ is infinite, and the phase angles - determine by the pole angles - are

$$\arg KL(s_1) = +90^\circ, \ \arg KL(s_2) = 0^\circ, \ \arg KL(s_3) = -90^\circ$$

- This shows that the Nyquist plot is completed by aan infinite arc to the right.
- Because the modified Nyquist path does not encircle the pole at the origin, we have $P = 0$, and from the Figure we conclude that there is no encirclement of -1 for any positive value of K (i.e. $N = 0$)
- Therefore the closed-loop system is stable for any positive gain.

The Nyquist Stability Criterion: Example



The Nyquist Stability Criterion: Example

Consider the transfer function

$$L(s) = \frac{10}{(s+1)^3} = \frac{10}{s^3 + 3s^2 + 3s + 1}$$

Substituting $j\omega$ for s gives

$$L(j\omega) = \frac{10}{(j\omega)^3 + 3(j\omega)^2 + 3j\omega + 1}$$

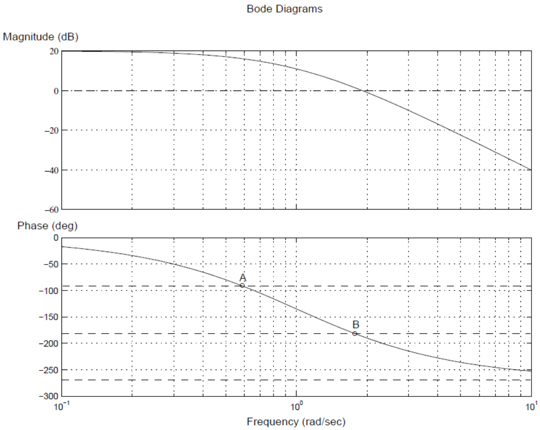
At low frequencies $\omega << 1$ we have

$$L \approx 10 \;\Rightarrow\; |L| \approx 10,\; \phi \approx 0^\circ$$

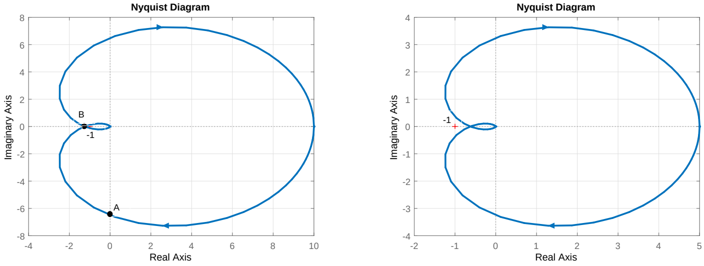
and at high frequencies

$$L \approx \frac{10}{(j\omega)^3} = \frac{10}{-j\omega^3} = j\frac{10}{\omega} \;\Rightarrow\; |L| \approx \frac{10}{\omega^3},\; \phi \approx -270^\circ$$

The Nyquist Stability Criterion: Example



The Nyquist Stability Criterion: Example

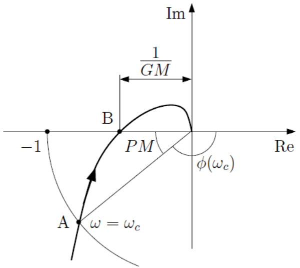


- The left hand side $K = 1$
- The right hand side $K = 0.5$
- The Nyquist plot of $L(s)$ encircles the critical point -1 two time clockwise $(N = 2)$. Because $L(s)$ has no unstable pole $(P = 0)$, this implies that the closed-loop system has two unstable poles when $K = 1$

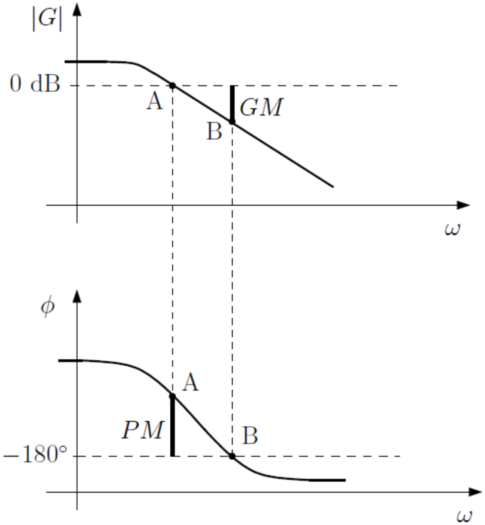
Gain and Phase Margin

The Nyquist stability test does not only indicate whether a closed-loop system is stable, it also gives information about the "distance" from the stability boundary.

- Such a measure is called a *stability margin*
- The upper bound on K, which the closed-loop system becomes unstable, is called the *gain margin* (GM) of the system.
- The *phase margin* (PM) is defined as the maximum phase change of $L(s)$ that the closed-loop system can tolerate before it becomes unstable.



Gain and Phase Margin



Reference

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2. Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini, "*Feedback Control of Dyanmic Systems*",4^{th} edition, Prentice Hall, 2002
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