INC 342 Feedback Control Systems: Lecture 10 Root Locus Design

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The steps for sketching a root locus

Step 1 Mark open-loop poles and zeros in the s-plane

Step 2 Mark real axis portion of the root locus to the left of an odd number of poles and zeros.

Step 3 Find asymptotes for the $n - m$ root locus branches that go to infinity:

$$\alpha = \frac{\sum_{i=1}^{n} \text{Re} p_i - \sum_{i=1}^{m} \text{Re} z_i}{n - m}, \ \phi_l^{\infty} = \frac{180^\circ}{n - m} + l\frac{360^\circ}{n - m}, \ l = 0, \ldots, n - m - 1$$

Step 4 Use the phase condition on a test point along a small circle around an open-loop pole to find the angle of departure:

$$\arg L(s) = \sum_{i=1}^{m} \arg(s - z_i) - \sum_{i=1}^{n} \arg(s - p_i) = 180^\circ + l360^\circ$$

Step 5 Compute breakaway and break-in points from

$$b\frac{da}{ds} - a\frac{db}{ds} = 0$$

The steps for sketching a root locus: Example

Consider an open-loop transfer function

$$L(s) = \frac{K}{s(s+4)(s+5)}$$

The system has three poles and on zero, so the angles of the three asymptotes can be calculated as

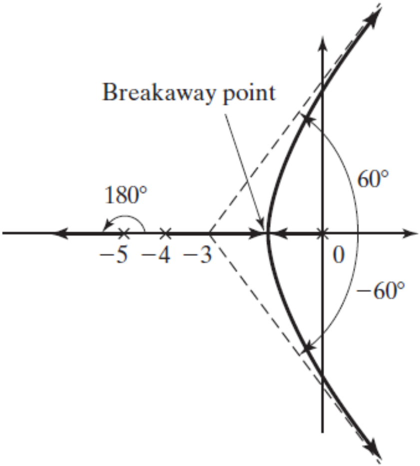
$$\theta_l^\infty = \frac{180^\circ}{3} + l\frac{360^\circ}{3} = 60^\circ, -60^\circ, 180^\circ$$

for $l = 0, -1$ and 1 and the intersection of the asymptotes with the real axis is given by

$$\alpha = \frac{0-4-5}{3} = -3.$$

The asymptotes clearly indicate that the system will become unstable when the gain is sufficiently large.

The steps for sketching a root locus: Example



The steps for sketching a root locus: Example

Consider an open-loop transfer function

$$L(s) = \frac{K(s^2+4s+8}{(s+3)(s^2+2s+2} = \frac{K(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}$$

with zeros at $z_{1,2} = -2 \pm j2$ and poles at $p_1 = -3$ and $p_{2,3} = -1 \pm j$
The departure angel ϕ at $p_2 = -1 + j$ satisfies the following equation

$$\angle(p_2 - z_1) + \angle(p_2 - z_2) - \angle(p_2 - p_1) - \phi - \angle(p_2 - p_3) = -180^\circ$$

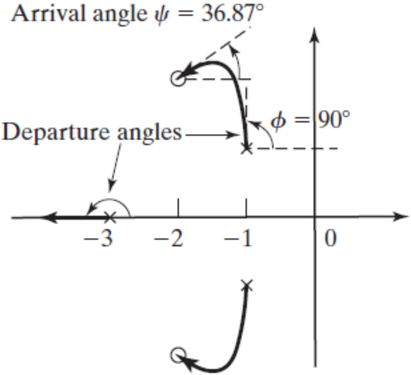
i.e. $-45^\circ + \tan^{-1} 3 - 90^\circ - \phi - \tan^{-1} \frac{1}{2} = -180^\circ$, then $\phi = 90^\circ$. and the arrival angle at the zero $z_1 = -2 + j2$ can be calculate from

$$\psi + \angle(z_1 - z_2) - \angle(z_1 - p_1) - \angle(z_1 - p_2) - \angle(z_1 - p_3) = -180^\circ$$

i.e.,

$$\psi + 90^\circ - \tan^{-1} 2 - 135^\circ - (180^\circ - \tan^{-1} 3) = -180^\circ \;\Rightarrow\; \psi = 36.87^\circ$$

The steps for sketching a root locus: Example



Not standard application

Consider a feedback system with

$$G(s)\,C(s) = \frac{4(s+3)}{s(s+1)(s+K)},$$

where K is a variable pole position. We would like to analyze how K effects the system stability and performance. The problem is obviously not in the standard root-locus format.

$$1 + G(s)\,C(s) = 0 \;\Rightarrow\; s(s+1)(s+K) + 4(s+3) = 0$$

which can be written as

$$(s+2)(s^2 - s + 6) + Ks(s+1) = 0 \;\Rightarrow\; 1 + \frac{Ks(s+1)}{(s+2)(s^2 - s + 6)} = 0$$

Then

$$L(s) = \frac{Ks(s+1)}{(s+2)(s^2 - s + 6)}$$

Not standard application

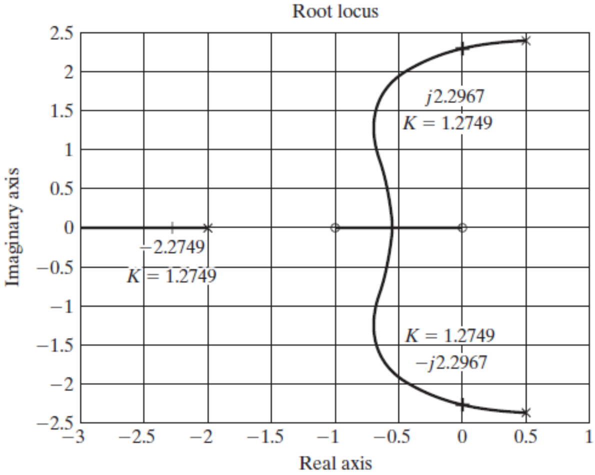
Matlab code

```
s = tf('s');
L = (s*(s+1))/((s+2)*(s^2 - s + 6));    % create the transfer function
rlocus(L)                                % generate a root locus
[K, poles] = rlocfind(L)
% to find the gain K and poles locations at the desired point.
```

Here we want to find the critical value of K where the root locus enters the left half plane (make closed-loop system unstable).

To do this by hand, we could use the Routh-Horwitz stability test to the characteristic polynomial:

Not standard application



Not standard application

Routh's criterion

Is the system with denominator polynomial

$$s(s+1)(s+K)+4(s+3)=s^3+(1+K)s^2+(K+4)s+12$$

Form the Routh array

$$\begin{array}{c|ccc} s^3 & 1 & K+4 & 0 \\ s^2 & 1+K & 12 & 0 \\ s^1 & c_1 & 0 & \\ s^0 & d_1 & & \end{array}$$

$$c_1 = \frac{-\begin{vmatrix} 1 & K+4 \\ 1+K & 12 \end{vmatrix}}{1+K}$$

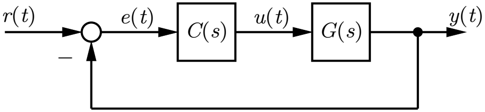
$$(K+1)(K+4)-12>0$$

$$K>\frac{\sqrt{57}-5}{2}=1.2749$$

- We don't need to calculate d_1
- Substituting the value of K to get the critical closed-loop poles.

Phase-lag controller

Consider a closed-loop system shown in Figure below

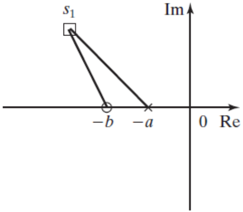


We shall consider a first-order phase-lag controller in the following form:

$$C(s) = \frac{K(s+b)}{s+a}, \; b > a > 0 \qquad C(0) = K\frac{b}{a}$$

- The phase-lag controller has the potential to increase the steady-state gain constant by b/a times compared to a pure gain controller.
- A phase-lag controller is designed in such a way that $C(s)$ contributes very little phase at the desired closed-loop pole locations while providing substantial gain increase to reduce the steady-state error.

Phase-lag controller



Phase-lag controller design

- Construct a root-locus plot of $KP(S)$
- Find the closed-loop (dominant) poles s_1 and $\bar{s}_1$ on the root -locus plot that will give the desired transient response and find the corresponding K value, say K_0
- Calculate the value of K required to yield the desired steady-state response and denote this K and K_s
- Pick a number $b(> a)$ that is much smaller than $|s_1|$ (so that $\frac{s_1+b}{s_1+a} \approx 1$) and let $a = K_0 b / K_s$
- Verify the controller design by simulation with $C(s) = \frac{K_0(s+b)}{s+a}$. Note that $C(0) = K_s$.
- For PI controller, $K_s = \infty$.

Phase-lag controller: example

Consider a unity feedback system with a plant model

$$G(s) = \frac{10}{s(s+5)(s+10)}$$

We would like to design a controller so that

- the response of the closed-loop system with respect to a step input has no more than 20% overshoot and the settling time is no greater than 4 sec;
- the steady-state error with respect to a ramp input is no more than 0.05.

Find the dominant pole locations. Consider a standard second-order system

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \; s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

From

$$\%M_P = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\% = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

Phase-lag controller: example

to guarantee that the overshoot is no more than 20%, we need the damping ratio to satisfy

$$\zeta \geq \frac{-\ln(0.2)}{\sqrt{\pi^2 + (\ln(0.2))^2}} = 0.456$$

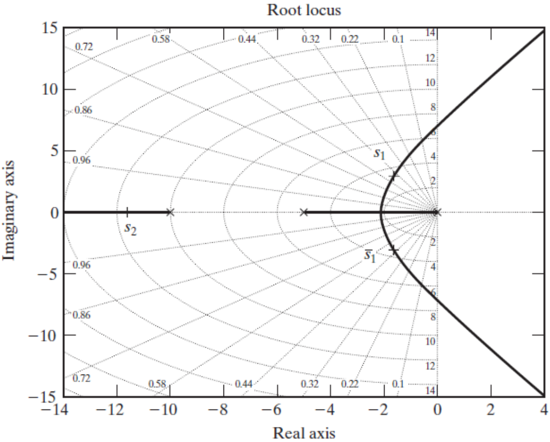
To guarantee that the settling time is no more than 4 sec with 2% tolerance, we need

$$\zeta\omega_n \geq \frac{4}{t_s} = 1$$

Phase-lag controller: example

Construct a root-locus plot for

$$L(s) = \frac{10K}{s(s+5)(s+10)}$$



Phase-lag controller: example

We need the real part of s_1 less than 1 and they have approximately the damping ratio 0.5 by using the Matlab command `[K0, s1] = rlocfind(L)` we then get

$$K_0 = 13,\ s_1 = -1665 \pm j2.8927,\ s_2 = -11.67$$

Find the required gain to satisfy the desired steady-state error. Note that $K_s = C(0)$ and

$$K_v = \lim_{s\to 0} sG(s)C(s) = \lim_{s\to 0} s\frac{10}{s(s+5)(s+10)}C(s) = \frac{C(0)}{5}$$

Thus

$$e_{ss} = \frac{1}{K_v} = \frac{5}{C(0)} \leq 0.05$$

gives $K_s = C(0) \geq 100$. Take $K_s = 100$.

Phase-lag controller: example

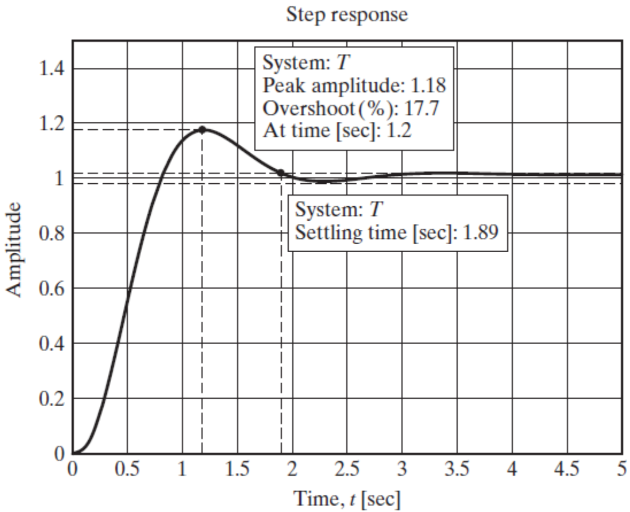
Take $b = 0.05$ which is much smaller than $|s_1|$. Then $a = K_0 b / K_s = 0.0065$ and we have a controller

$$C(s) = \frac{13(s + 0.05)}{s + 0.0065}$$

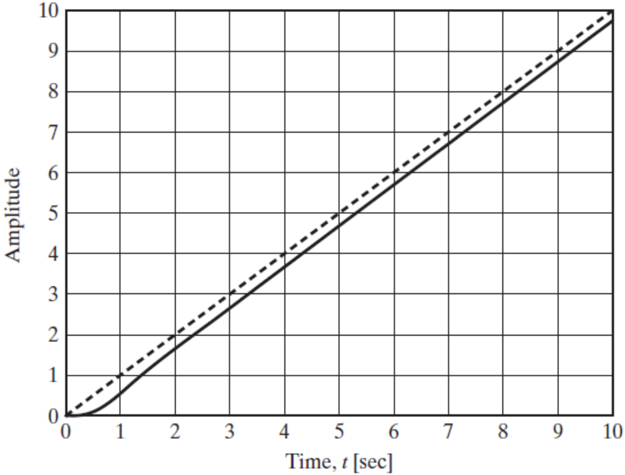
which gives the closed-loop poles at

$$-11.6656, -0.0509, -1.6450 \pm j2.8724$$

Phase-lag controller: example



Phase-lag controller: example



Phase-lead controller

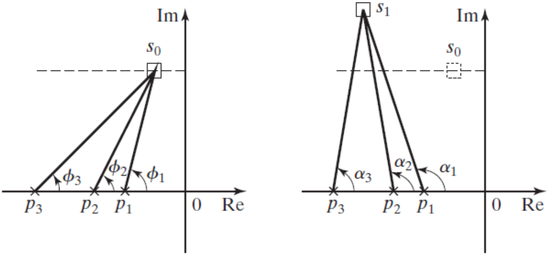
From the previous example, if it is required that the settling time be less than 1 sec. It is fairly easy to see that no phase-lag (or PI) controller can satisfy this specification since it requires that the dominant poles satisfy $\text{Re}\{s_1\} \leq -4$. We need a phase-lead (or PID) controller to move the dominant poles a little further away from the imaginary axis.

A first-order phase-lead controller has the following general form:

$$C(s) = \frac{K(s+b)}{s+a}, \; a > b > 0$$

Since $\angle C(s) > 0$ for any s on the upper half of the comple plane, it contributes a positive angle (or phase-lead).

Phase-lead controller



Consider a system with three poles p_1, p_2, and p_3. Suppose s_0 is a point on the root locus

$$-\phi_1 - \phi_2 - \phi_3 = -180^\circ$$

We need to move the closed-loop pole from s_0 to s_1; then we would need the following phase lead to guarantee that s_1 point satisfies the phase condition:

$$\theta := \alpha_1 + \alpha_2 + \alpha_3 - \phi_1 - \phi_2 - \phi_3 > 0$$

Phase-lead controller

Phase-lag controller design

Step 1 Construct a root-locus plot of $KG(s)$.

Step 2 Determine the desired closed-loop (dominant) poles s_1 and $\bar{s}_1$ that will give the desired transient response.

Step 3 Calculate the angel required so that s_1 is on the root-locus plot

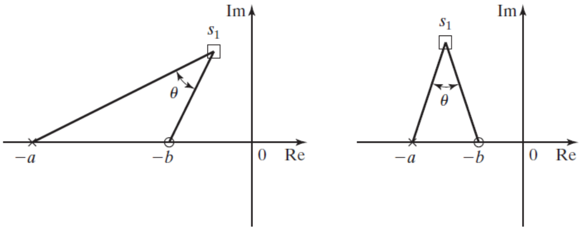
$$\angle C(s_1) + \angle G(s_1) = (2k+1)180^\circ,$$
$$\theta = \angle C(s_1) = (2k+1)180^\circ - \angle G(s_1) > 0$$

Step 4 Find b and a so that

$$\angle(s_1 + b) - \angle(s_1 + a) = \theta$$

and make sure that s_1 is the dominant pole (Note that there are infinitely many choices.)

Phase-lead controller



Step 5 Find K_0 so that

$$\frac{K_0|s_1 + b|}{|s_1 + a|}|G(s_1)| = 1$$

Step 6 Verify the controller design by simulation with

$$C(s) = \frac{K_0(s + b)}{s + a}$$

Phase-lead controller: example

Consider a system

$$G(s) = \frac{10}{s(s+5)(s+10)}$$

We would like to design a controller so that

- the response of the closed-loop system with respect to a step input has no more than 20% overshoot and the settling time is no more than 1 sec.

As in the previous example, we can find the desired dominant pole locations:

$$\zeta \geq 0.456, \ \zeta\omega_n \geq 4$$

and from the previous example, it is clear that a phase-lag or PI controller cannot satisfy these design specifications. Using phase-lead compensator, we pick the dominant poles conservatively at

$$s_1 = -5 + j5, \ \bar{s}_1 = -5 - j5$$

Then $G(s_1) = j0.04$ by hand or using `evalfr(G,s1)` and $\angle G(s_1) = 90°$.

Phase-lead controller: example

Thus, we need

$$\theta = \angle C(s_1) = 180^\circ - 90^\circ = 90^\circ$$

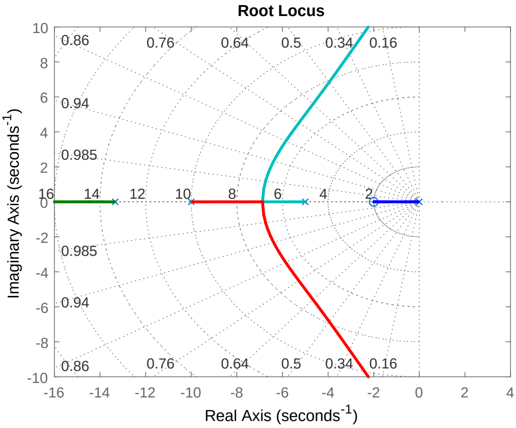
To pick b and a, if $b \geq 5$, then a must be ∞, since $\theta = 90^\circ$. As a first try, we pick $b = 2$. Then, a can be solved from

$$\angle(s_1 + b) - \angle(s_1 + a) = \theta$$

which gives $a = 13.3333$. Now K_0 can be found as $K_0 = 41.667$. Thus, we have a phase-lead controller

$$C_0(s) = \frac{41.667(s + 2)}{s + 13.333}$$

Phase-lead controller: example



Phase-lead controller: example

The closed-loop poles are (using a command `pole(feedback(G*C_0,1))`)

$$-17.3739, \ -5 \pm j5, \ -0.9593$$

and the pair of poles at $-5 \pm j5$ are actually not dominant poles. The pole at -0.9593, which is not close to any zero, turns out to be the dominant poles.
We shall try with $b = 4$. Then, we have $a = 30$ and $K_0 = 125$, which give

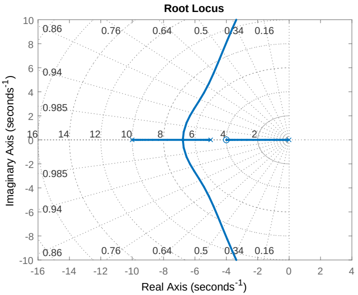
$$C_1(s) = \frac{125(s+4)}{s+30}$$

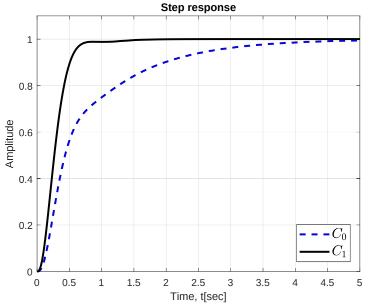
The closed-loop poles at

$$-31.8614, \ -5 \pm j5, \ -3.1386$$

The poles $-5 \pm j5$ are also not the dominant poles but the dominant pole at -3.1386 is not very close to the closed-loop zero at -4. The design meets the requirement by looking at the step response.

Phase-lead controller: example





The step response shown in Fig above indicates that the design specifications are met the requirements. The velocity constant can be calculated as $K_v = 10/3$ with $C_1(s)$. Thus the steady-state error with respect to ta ramp input is $e_{ss} = 1/K_v = 3/10 = 0.3$.

Phase-lead controller: example

It is sometimes more convenient to first pick some controller parameters. For example, one can first choose the desired controller zero and pole, and then determine and appropriate gain. Using the same example we have

- the open loop pole at -5 in the transfer function $G(s)$ tends to pull the closed-loop poles to the right half plane.
- It is appropriate to place the zero of the lead controller at the same location to cancel the effect of that pole.

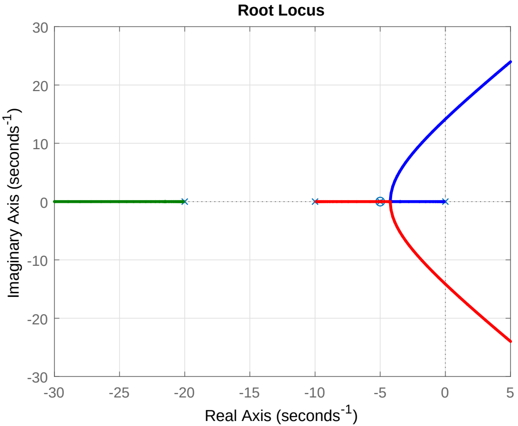
Then, we choose $b = 5$ and then $a = 20$.

$$C_2(s) = \frac{K(s+5)}{s+20}$$

Then

$$L(s) = C(s)G(s) = \frac{10K}{s(s+10)(s+20)}$$

Phase-lead controller: example



Choosing a suitable gain K from the root locus, we obtain the lead controller,

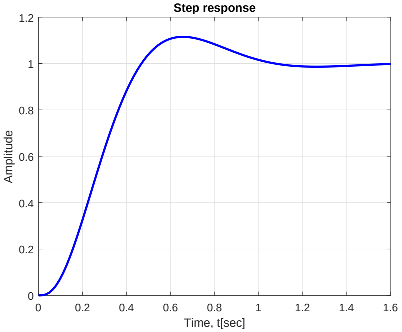
$$C(s) = \frac{90(s+5)}{s+20}$$

Phase-lead controller: example

The closed-loop poles with this controller are given by

$$-23.0074, \ -3.4963 \pm j5.1859$$

The step response with the controller shown in Fig indicates that the design specifications are met.



Lead-Lag compensator

- Phase-lag or Phase-lead controller may not provide enough design freedom to satisfy design specifications.
- A combination of phase-lag and phase-lead (or PID) controller is necessary.
- The basic idea is to use a phase-lead to satisfy the transient response and a phase-lag to satisfy the steady-state requirement.

Lead-Lag compensator: Example

Consider a unity feedback system with

$$G(s) = \frac{10}{s(s+5)(s+10)}$$

We would like to design a controller so that

- the response of the closed-loop system with respect to a step input has no more than 20% overshoot and the settling time is no greater than 1 sec;
- the steady-state error with respect to a ramp input is no more than 0.05.

From the previous example the phase-lead controller is

$$C_{\text{lead}}(s) = \frac{125(s+4)}{s+30}$$

will satisfy the transient specifications, i.e. PO $\leq 20\%$ and $t_s \leq 1$ sec. The new plant model is

$$G_{\text{lead}}(s) = G(s)\,C_{\text{lead}}(s) = \frac{1250(s+4)}{s(s+5)(s+10)(s+30)}$$

Lead-Lag compensator: Example

We shall now design a phase-lag controller for $G_{\text{lead}}(s)$ so that the steady-state error specification is satisfied. The steady-state error specification requires $K_v \geq 20$. We shall take $K_v = 20$.
The desired closed-loop poles at

$$-31.8614, \ -5 \pm j5, \ -3.1386$$

which are the closed-loop poles with the phase-lead controller $C_{\text{lead}}(s)$. We have $K_v = 10/3$ when only the phase-lead controller is used. Then we need to increase the system gain by

$$K_s = \frac{20}{10/3} = 6$$

With a trial-and-error method, we pick $b = 0.08$ (at least 10 times smaller than the dominant poles). Then $a = b/K_s = 0.08/6 = 0.01333$ (with $K_0 = 1$). Thus we have

$$C_{\text{lag}}(s) = \frac{s+b}{s+a} = \frac{s+0.08}{s+0.01333}$$

Lead-Lag compensator: Example

and finally the desired lead-lag controller is

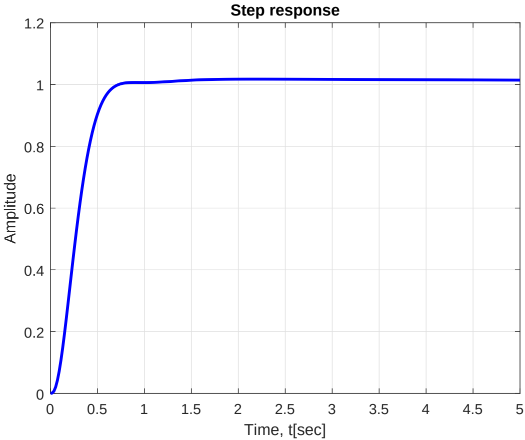
$$C(s) = C_{\text{lead}}(s)C_{\text{lag}}(s) \qquad = \frac{125(s+4)}{s+30}\frac{s+0.08}{s+0.01333}$$

The closed-loop poles with this lead-lag controller are at

$$-31.858,\ -4.9821 \pm j4.9624,\ -3.1094,\ -0.0817$$

Note that the closed-loop pole at -0.0817 is almost canceled by a closed-loop zero at -0.08, and therefore they will have almost no effect on system response.

Phase-lead controller: example



Delay-Time System

To use root-locus design method with a first-order plus dead-time system

$$G(s) = G_0(s)e^{-T_d s}$$

One way we can approximate the delay-time with a **Padé approximation**. The idea is to develop both $e^{-T_d s}$ and the rational function into a series and match a suitable number of terms. We have

$$e^{-T_d s} = 1 - T_d s + \frac{(T_d s)^2}{2!} - \frac{(T_d s)^3}{3!} + \cdots$$

and choosing a first order rational function

$$\frac{\beta_1 T_d s + \beta_0}{\alpha_1 T_d s + 1} = \beta_0 + (\beta_1 - \beta_0\alpha_1) T_d s + (\ldots)(T_d s)^2 + \ldots$$

values α_1, β_0, and β_1 can be obtained by comparing the coefficients of both series.

Reference

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