Fourier Transform II

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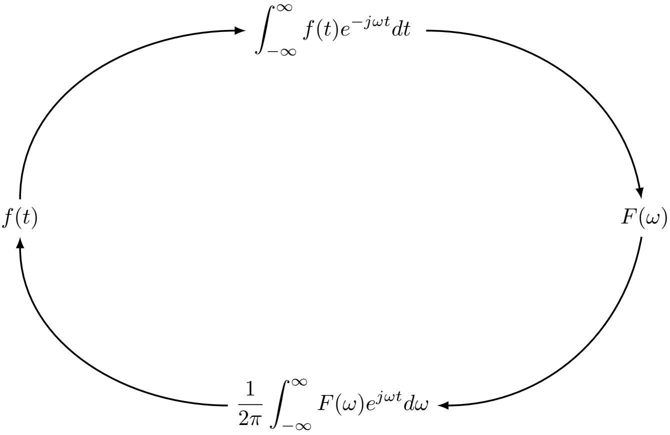
Some properties of the Fourier Transform

the time-frequency duality



The direct and the inverse Fourier transforms.

Some properties of the Fourier Transform

the time-frequency duality

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

- the direct and the inverse transform operations are remarkably similar.
- the factor 2π appears only in the inverse operator, and the exponential indices in the two operations have opposite signs. Otherwise the two operations are symmetrical.
- It is the basis of the so-called duality of time and frequency.
- For example, the time-shifting property, to be proved later, states that if $f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$, then

$$f(t-t_0) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)e^{-j\omega t_0}$$

$$f(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega-\omega_0)$$

- This properties of the Fourier transform are useful not only in deriving the direct and inverse transforms of many functions, but also very useful in signal processing.

Symmetry Property

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$$

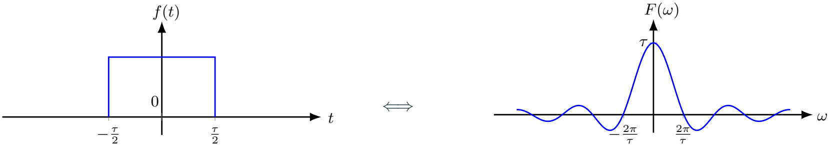
then

$$F(t) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi f(-\omega)$$

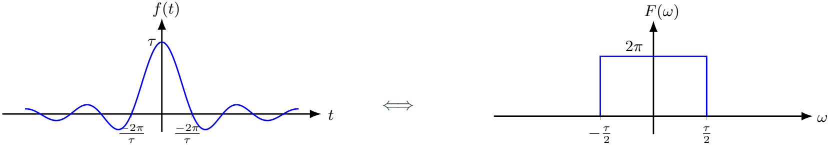
Proof:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{jxt} dx$$
$$2\pi f(-t) = \int_{-\infty}^{\infty} F(x) e^{-jxt} dx$$

Changing t to ω yields the result.



(a)



(b)

We have

$$\underbrace{\operatorname{rect}\left(\frac{t}{\tau}\right)}_{f(t)} \overset{\mathcal{F}}{\Longleftrightarrow} \underbrace{\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)}_{F(\omega)}$$

Here $F(t)$ is the same as $F(\omega)$ with ω replaced by t, and $f(-\omega)$ is the same as $f(t)$ with t replaced by $-\omega$. Using the symmetry property yields

$$\underbrace{\tau \operatorname{sinc}\left(\frac{\tau t}{2}\right)}_{F(t)} \overset{\mathcal{F}}{\Longleftrightarrow} \underbrace{2\pi \operatorname{rect}\left(\frac{-\omega}{\tau}\right)}_{2\pi f(-\omega)} = 2\pi \operatorname{rect}\left(\frac{\omega}{\tau}\right)$$

Note that $\operatorname{rect}(-x) = \operatorname{rect}(x)$ because rect is an even function.

Some properties of the Fourier Transform

Symmetry Property :examples II

Show that $\frac{1}{jt+a} \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi e^{a\omega}u(-\omega)$

Solution:

$$\underbrace{e^{-at}u(t)}_{f(t)} \overset{\mathcal{F}}{\Longleftrightarrow} \underbrace{\frac{1}{j\omega + a}}_{F(\omega)}$$

$$\underbrace{\frac{1}{jt + a}}_{F(t),(\omega\to t)} \overset{\mathcal{F}}{\Longleftrightarrow} \underbrace{2\pi e^{a\omega}u(-\omega)}_{2\pi f(-\omega),(t\to -\omega)}$$

Show that $\delta(t + t_0) + \delta(t - t_0) \overset{\mathcal{F}}{\Longleftrightarrow} 2\cos t_0\omega$

Solution:

$$\cos\omega_0 t \overset{\mathcal{F}}{\Longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\pi[\delta(t + t_0) + \delta(t - t_0)] \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi\cos t_0(-\omega)$$

Scaling Property

If

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$$

then, for any real constant a,

$$f(at) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Proof:

$$\mathcal{F}\left[f(at)\right] = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt = \frac{1}{a}\int_{-\infty}^{\infty} f(x) e^{(-j\omega/a)x} dx$$

Let $\tau = at$, for $a > 0$, we have $t = \tau/a$, $t = \infty \rightarrow \tau = \infty$, $t = -\infty \rightarrow \tau = -\infty$

$$\mathcal{F}\left\{f(at)\right\} = \frac{1}{a}\int_{-\infty}^{\infty} f(\tau) e^{(-j\omega/a)\tau} d\tau = \frac{1}{a} F(\frac{\omega}{a})$$

Some properties of the Fourier Transform

Scaling Property

If $a < 0$, the integration limits flip which introduces an extra minus sign, we have $t = \infty \rightarrow \tau = -\infty$, $t = -\infty \rightarrow \tau = \infty$

$$\begin{aligned}\mathcal{F}\left\{f(at)\right\} &= \frac{1}{a}\int_{\infty}^{-\infty} f(\tau)e^{(-j\omega/a)\tau}d\tau = -\frac{1}{a}\int_{-\infty}^{\infty} f(\tau)e^{(-j\omega/a)\tau}d\tau \\ &= -\frac{1}{a}F\left(\frac{\omega}{a}\right)\end{aligned}$$

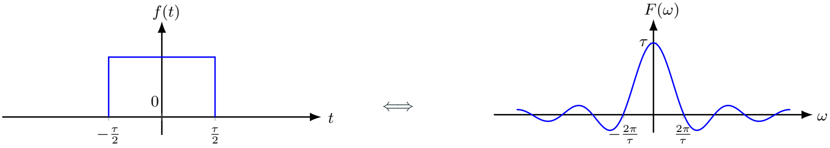
Hence,

$$\begin{aligned}F(\omega) &= \begin{cases}\frac{1}{a}F\left(\frac{\omega}{a}\right), & a > 0 \\ -\frac{1}{a}F\left(\frac{\omega}{a}\right), & a < 0\end{cases} \\ &= \frac{1}{|a|}F\left(\frac{\omega}{a}\right)\end{aligned}$$

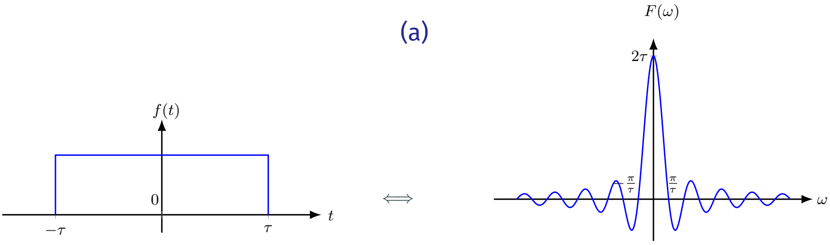
Some properties of the Fourier Transform

Scaling Property

- The function $f(at)$ represents the function $f(t)$ compressed in time by a factor a.
- Similarly, a function $F(\omega/a)$ represents the function $F(\omega)$ expanded in frequency by the same factor a.
- for example $\cos 2\omega_0 t$ is the same as the signal $\cos \omega_0 t$ time-compressed by a factor of 2. Clearly, the spectrum of the former (impulse at $\pm 2\omega_0$) is an expanded version of the spectrum of the latter (impulse at $\pm\omega_0$).
- The scaling property implies that if $f(t)$ is wider, its spectrum is narrower, and vice versa.
- Doubling the signal duration halves its bandwidth and vice versa.



(a)



(b)

Time and Frequency inversion

From,

$$f(at) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

By letting $a = -1$, we obtain the **time and frequency inversion property**

$$f(-t) \overset{\mathcal{F}}{\Longleftrightarrow} F(-\omega)$$

Example Find the Fourier transforms of $e^{at}u(-t)$ and $e^{-a|t|}$. From

$$e^{-at}u(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{j\omega + a} \text{ then } \underbrace{e^{at}u(-t)}_{f(-t)} \overset{\mathcal{F}}{\Longleftrightarrow} \underbrace{\frac{1}{-j\omega + a}}_{F(-\omega)}$$

Then

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$e^{-a|t|} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{j\omega + a} + \frac{1}{-j\omega + 1} = \frac{2a}{a^2 + \omega^2}$$

Some properties of the Fourier Transform

Time and frequency inversion property

Time-Shifting

If

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega) \quad \text{then} \quad f(t-t_0) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)e^{-j\omega t_0}$$

Proof: By definition,

$$\mathcal{F}\left[f(t-t_0)\right] = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt$$

Letting $t - t_0 = x$, we have

$$\begin{aligned}\mathcal{F}\left[f(t-t_0)\right] &= \int_{-\infty}^{\infty} f(x)e^{-j\omega(x+t_0)}dx \\ &= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx = F(\omega)e^{-j\omega t_0}\end{aligned}$$

The result shows that delaying a signal by t_0 seconds does not change its amplitude spectrum. The phase spectrum, however, is changed by $-\omega t_0$.

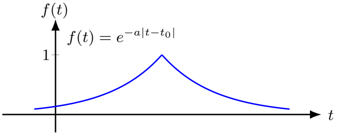
Some properties of the Fourier Transform

Physical Explanation of the Linear Phase

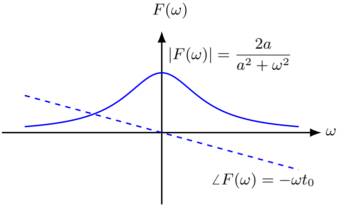
- time delay in a signal causes a linear phase shift in its spectrum but does not change its amplitude spectrum.

Example Find the Fourier transform of $e^{-a|t-t_0|}$
This is the time-shift version of $e^{-a|t|}$

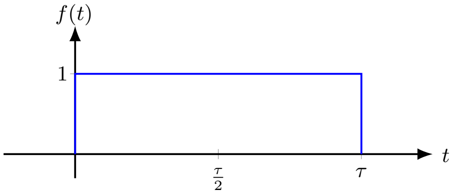
$$e^{-a|t-t_0|} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{2a}{a^2+\omega^2} e^{-j\omega t_0}$$



$\Longleftrightarrow$



Find the Fourier transform of the gate pulse $f(t)$



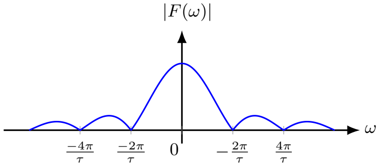
The pulse $f(t)$ is the gate pulse $\mathrm{rect}\left(\frac{t}{\tau}\right)$ delayed by $\tau/2$ seconds. Hence, its Fourier transform is the Fourier transform of $\mathrm{rect}\left(\frac{t}{\tau}\right)$ multiplied by $e^{-j\omega\tau/2}$. Therefore

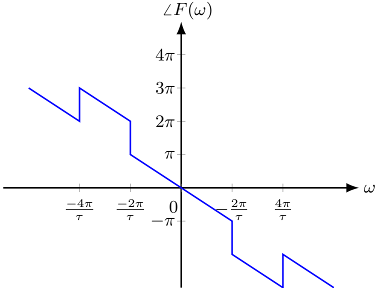
$$F(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)e^{-j\omega\tau/2}$$

The amplitude spectrum $|F(\omega)|$ of this pulse is the same as $\mathrm{rect}\left(\frac{t}{\tau}\right)$. But the phase spectrum has an added linear term $-\omega\tau/2$.

Some properties of the Fourier Transform

Physical Explanation of the Linear Phase: Example





Frequency-Shifting

If

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega) \quad \text{then} \quad f(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega - \omega_0)$$

Proof: By definition,

$$\begin{aligned} \mathcal{F}\left[f(t)e^{j\omega_0 t}\right] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt = F(\omega - \omega_0) \end{aligned}$$

Changing ω_0 to $-\omega_0$ this property yields

$$f(t)e^{-j\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega + \omega_0)$$

This property shows that the multiplication of a signal by a factor $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$. Note the duality between the time-shifting and the frequency-shifting properties.

Some properties of the Fourier Transform

The Frequency-Shifting Property

Because $e^{j\omega_0 t}$ is not a real function that can be generated, frequency shifting in practice is achieved by multiplying $f(t)$ by sinusoid.

$$f(t)\cos\omega_0 t = \frac{1}{2}\left[f(t)e^{j\omega_0 t} + f(t)e^{-j\omega_0 t}\right]$$

It follows that

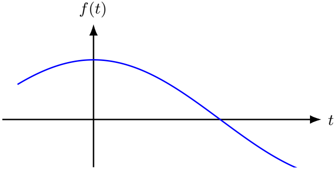
$$f(t)\cos\omega_0 t \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right]$$

The result shows that the multiplication of a signal $f(t)$ by a sinusoid of frequency ω_0 shifts the spectrum $F(\omega)$ by $\pm\omega_0$

- Multiplication of a sinusoid $\cos\omega_0 t$ by $f(t)$ amounts to modulation the sinusoid amplitude.
- This type of modulation is known as **amplitude modulation.**
- The sinusoid $\cos\omega_0 t$ is called the **carrier**, the signal $f(t)$ is the **modulating signal**, and the signal $f(t)\cos\omega_0 t$ is the **modulated signal**.

Some properties of the Fourier Transform

The Frequency-Shifting Property



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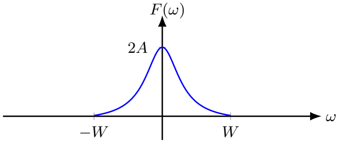
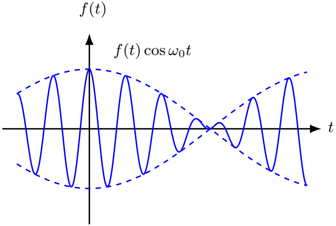
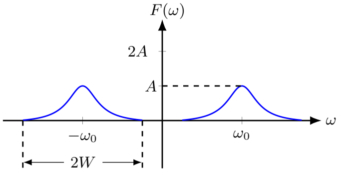




$\Longleftrightarrow$



Find and sketch the Fourier transform fo the modulated signal $f(t)\cos 10t$ in which $f(t)$ is a gate pulse $\text{rect}\left(\frac{t}{4}\right)$
Solution: From Fourier Transform table, we have

$$\text{rect}\left(\frac{t}{4}\right) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad 4\,\text{sinc}\,(2\omega)$$

It follows that

$$f(t)\cos 10t \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{2}\left[F(\omega+10)+F(\omega-10)\right]$$

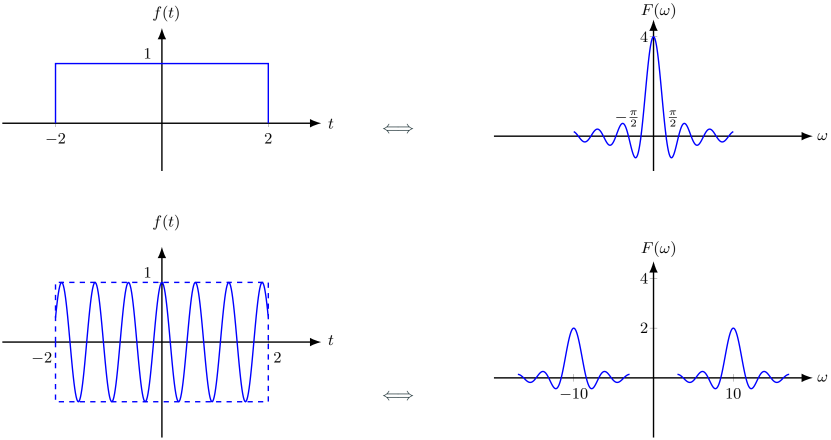
In this case, $F(\omega) = 4\,\text{sinc}(2\omega)$, Therefore

$$f(t)\cos 10t \;\overset{\mathcal{F}}{\Longleftrightarrow}\; 2\,\text{sinc}[2(\omega+10)] + 2\,\text{sinc}[2(\omega-10)]$$

The spectrum of $f(t)\cos 10t$ is obtained by shifting $F(\omega)$ to the left by 10 and also to the right by 10, and then multiplying it by one-half.

Some properties of the Fourier Transform

The Frequency-Shifting Property: Example



Some properties of the Fourier Transform

The Frequency-Shifting Property: Application

- Modulation is used to shift signal spectra.
- If several signals, each occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere; it will be impossible to separate or retrieve them at a receiver. For example, if all radio stations decide to broadcast audio signals simultaneously, the receiver will not be able to separate them.
- This problem is solved by using modulation, whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal. This procedure shifts the signal spectrum to its allocated band, which is not occupied by any other station.
- A radio receiver can pick up any station by tuning to the band of the desired station. The receiver must now demodulate the received signal. Demodulation therefore consists of another spectral shift required to restore the signal to its original band.
- Both modulation and demodulation implement spectral shifting.
- The method of transmitting several signals simultaneously over a channel by sharing its frequency band is known as **frequency-division multiplexing (FDM)**

Some properties of the Fourier Transform

The Frequency-Shifting Property: Application

- For effective radiation of power over a radio link, the antenna size must be of the order of the wavelength of the signal to be radiated.
- Audio signal frequencies are so low (wavelengths are so large) that impractically large antennas will be required for radiation.
- Here, shifting the spectrum to a higher frequency (a smaller wavelength) by modulation solves the problem.

The time convolution property and its dual, the frequency convolution property, state that if

$$f_1(t) \overset{\mathcal{F}}{\Longleftrightarrow} F_1(\omega) \quad \text{and} \quad f_2(t) \overset{\mathcal{F}}{\Longleftrightarrow} F_2(\omega)$$

then **(time convolution)**

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\Longleftrightarrow} F_1(\omega)F_2(\omega)$$

and **(frequency convolution)**

$$f_1(t)f_2(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2\pi}F_1(\omega) * F_2(\omega)$$

Proof: By definition

$$\begin{aligned}\mathcal{F}\left[f_1(t) * f_2(t)\right] &= \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right]dt \\ &= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j\omega t}f_2(t-\tau)dt\right]d\tau\end{aligned}$$

The inner integral is the Fourier transform of $f_2(t-\tau)$, given by [time-shifting property] $F_2(\omega)e^{-j\omega\tau}$. Hence

$$\mathcal{F}\left[f_1(t) * f_2(t)\right] = \int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}F_2(\omega)d\tau = F_2(\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau = F_1(\omega)F_2(\omega)$$

The transfer function $H(\omega)$ is the Fourier transform of the unit impulse response $h(t)$. Thus

$$h(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad H(\omega)$$

Application of the time convolution property to $y(t) = f(t) * h(t)$ yields

$$Y(\omega) = F(\omega)H(\omega)$$

The frequency convolution property can be proved in exactly the same way by reversing the roles of $f(t)$ and $F(\omega)$.

Some properties of the Fourier Transform

Convolution : Example

Using the time convolution property, show that if

$$f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$$

then

$$\int_{-\infty}^{t} f(\tau)d\tau \overset{\mathcal{F}}{\Longleftrightarrow} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

Because

$$u(t-\tau) = \begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases}$$

it follows that

$$f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau$$

Now, from the time convolution property, it follows that

$$
\begin{aligned}
f(t) * u(t) = \int_{-\infty}^{t} f(\tau)d\tau \quad &\stackrel{\mathcal{F}}{\Longleftrightarrow} \quad F(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] \\
&= \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)
\end{aligned}
$$

Differentiation

If $f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$ then **(time differentiation)**

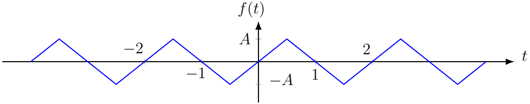
$$\frac{df}{dt} \overset{\mathcal{F}}{\Longleftrightarrow} j\omega F(\omega)$$

Proof: Differentiation of both side of the inverse Fourier transform equation as follow:

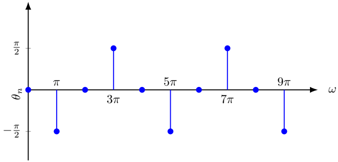
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$
$$\frac{df}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega)e^{j\omega t}d\omega$$

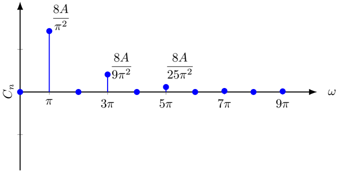
The result shows that $\frac{df}{dt} \overset{\mathcal{F}}{\Longleftrightarrow} j\omega F(\omega)$ and also $\frac{d^n f}{dt^n} \overset{\mathcal{F}}{\Longleftrightarrow} (j\omega)^n F(\omega)$

Using the time-differentiation property, find the Fourier transform of the triangle pulse $\Delta\left(\frac{t}{\tau}\right)$ illustrated in Figure below.



Solution: To find the Fourier transform of this pulse we differentiate the pulse successively 2 times.





The 2^{nd} derivative is

$$\frac{d^2 f}{dt^2} = \frac{2}{\tau}\left[\delta\left(t+\frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t-\frac{\tau}{2}\right)\right]$$

From the time-differentiation property

$$\frac{d^2 f}{dt^2} \overset{\mathcal{F}}{\Longleftrightarrow} (j\omega)^2 F(\omega) = -\omega^2 F(\omega)$$

From the time-shifting property

$$\delta(t-t_0) \overset{\mathcal{F}}{\Longleftrightarrow} e^{-j\omega t_0}$$

Then

$$-\omega^2 F(\omega) = \frac{2}{\tau}\left[e^{j\frac{\omega\tau}{2}} - 2 + e^{-j\frac{\omega\tau}{w}}\right] = \frac{4}{\tau}\left(\cos\frac{\omega\tau}{2} - 1\right) = -\frac{8}{\tau}\sin^2\left(\frac{\omega\tau}{4}\right)$$

$$F(\omega) = \frac{8}{\omega^2\tau}\sin^2\left(\frac{\omega\tau}{4}\right) = \frac{\tau}{2}\left[\frac{\sin\left(\frac{\omega\tau}{4}\right)}{\frac{\omega\tau}{4}}\right]^2 = \frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\omega\tau}{4}\right)$$

Integration

If $f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega)$ then **time integration**

$$\int_{-\infty}^{t} f(\tau)d\tau \overset{\mathcal{F}}{\Longleftrightarrow} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

Proof Because

$$u(t-\tau) = \begin{cases} 1, & \tau \leq t \\ 0, & \tau > t \end{cases}$$

It follows that

$$f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau$$

Some properties of the Fourier Transform

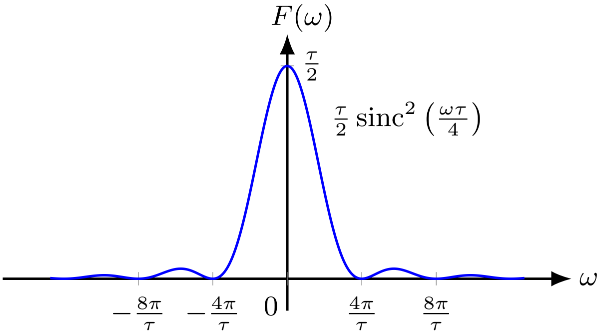
Time Integration

Now, from the time convolution property, it follows that

$$f(t) * u(t) = \int_{-\infty}^{t} f(\tau) d\tau \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F(\omega) \left[\frac{1}{j\omega} + \pi\delta(\omega) \right]$$

$$= \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

This procedure of finding the Fourier transform can be applied to any function $f(t)$ made up of straight-line segments with $f(t) \to 0$ as $|t| \to \infty$. This example suggests a numerical method of finding the Fourier transform of an arbitrary signal $f(t)$ by approximating the signal by straight-line segments. The spectrum of this example is show below.



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