Fourier Transform I

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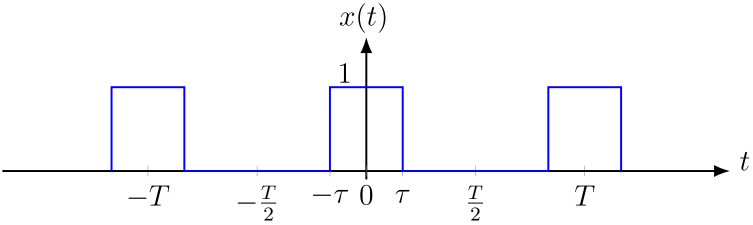


Find the exponential Fourier coefficients of a periodic rectangular pulse train with period T_0 . During the period $-\frac{T_0}{2} < t < \frac{T_0}{2}$ the pulse is given by

$$x(t) = \begin{cases} 1, & |t| < \tau \\ 0, & \tau < |t| < \frac{T_0}{2} \end{cases}$$

From ($\omega_0 = 2\pi/T_0$)

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

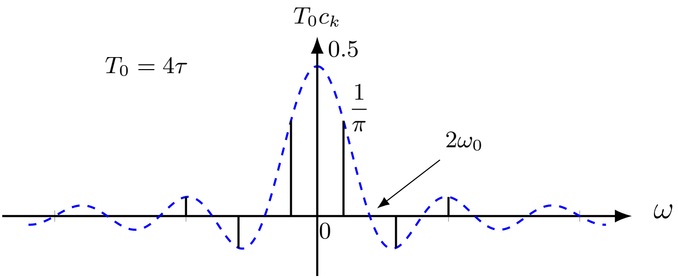
Introduction

Using an even function property of $x(t)$, we have $b_n = 0$

$$
\begin{aligned}
c_0 &= \frac{1}{T_0}\int_{-\tau}^{\tau} dt = \frac{1}{T_0}2\tau = \frac{2\tau}{T_0} \\
c_k &= \frac{1}{T_0}\int_{-\tau}^{\tau} e^{-jk\omega_0 t} dt = \frac{-1}{jk\omega_0 T_0} e^{-jk\omega_0 t}\Big|_{-\tau}^{\tau} \\
&= \frac{-1}{jk\omega_0 T_0}\left[\frac{e^{-jk\omega_0\tau} - e^{jk\omega_0\tau}}{2j}\right] = \frac{2}{k\omega_0 T_0}\sin k\omega_0\tau \\
&= \frac{\sin 2k\pi\frac{\tau}{T_0}}{k\pi}, \qquad k \neq 0 \text{ and } \omega_0 T_0 = 2\pi
\end{aligned}
$$

If we fix τ and vary T_0. For example, $T_0 = 4\tau$

$$
\begin{aligned}
c_k &= \frac{2\sin k\omega_0\tau}{k\omega_0 T_0} = \frac{\sin k\pi/2}{k\pi} \\
c_1 &= c_{-1} = \frac{1}{\pi}, \quad c_2 = c_{-2} = 0, \quad c_3 = c_{-3} = \frac{-1}{3\pi}, \quad c_4 = c_{-4} = 0 \\
c_5 &= c_{-5} = \frac{1}{5\pi}, \quad c_6 = c_{-6} = 0, \quad c_7 = c_{-7} = \frac{-1}{7\pi}, \quad c_8 = c_{-8} = 0
\end{aligned}
$$

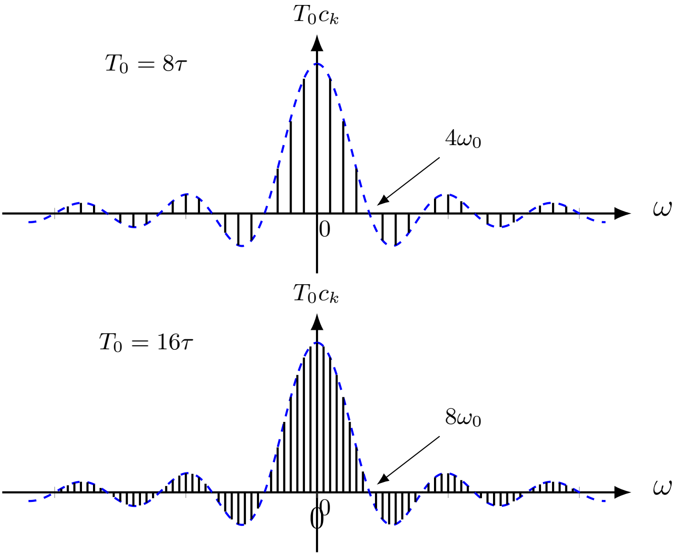


Consider that

$$c_k = \frac{2\sin k\omega_0\tau}{k\omega_0 T_0}$$

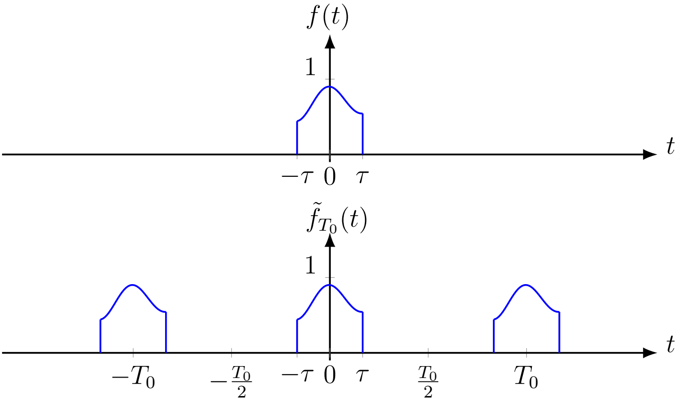
$$T_0 c_k = \frac{2\sin k\omega_0\tau}{k\omega_0} = \left.\frac{2\sin \omega\tau}{\omega}\right|_{\omega=k\omega_0}$$

We can think of that ω is a continuous variable. The function $\frac{2\sin\omega\tau}{\omega}$ represents the envelope of $T_0 c_k$ and these coefficients are simply equally spaced samples of this envelope.



$T_0 c_k$
$T_0 = 8\tau$
$4\omega_0$
ω
0
$T_0 c_k$
$T_0 = 16\tau$
$8\omega_0$
ω
0

Aperiodic Signal Representation by Fourier Integral



- We can construct a periodic signal $\tilde{f}(t)$ formed by repeating the signal $f(t)$ at intervals of T_0 second.
- The period T_0 is made long enough to avoid overlap between the repeating pulses.

Aperiodic Signal Representation by Fourier Integral

- The periodic signal $\tilde{f}(t)$ can be represented by an exponential Fourier series.
- If we let $T_0 \to \infty$, the pulses in the periodic signal repeat after an infinite interval and, therefore

$$\lim_{T_0 \to \infty} \tilde{f}(t) = f(t)$$

- Thus, the Fourier series representing $\tilde{f}(t)$ will also represent $f(t)$ in the limit $T_0 \to \infty$.
- The exponential Fourier series for $\tilde{f}(t)$ is given by

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{f}(t) e^{-jk\omega_0 t} dt, \qquad \omega_0 = \frac{2\pi}{T_0}$$

Aperiodic Signal Representation by Fourier Integral

- Since $\tilde{f}(t) = f(t)$ for $|t| < T_0/2$ and also $f(t) = 0$ outside this interval, then the above equation can be rewritten as

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0} \int_{-\infty}^{\infty} f(t) e^{-jk\omega_0 t} dt$$

- The envelope of $T_0 D_n$, the function of $\omega = n\omega_0$ can be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \qquad c_k = \frac{1}{T_0} F(k\omega_0)$$

- Since

$$\begin{aligned} \tilde{f}(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T_0} F(k\omega_0) e^{j\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(k\omega_0) e^{jk\omega_0 t} \omega_0, \qquad \omega_0 = \frac{2\pi}{T_0} \end{aligned}$$

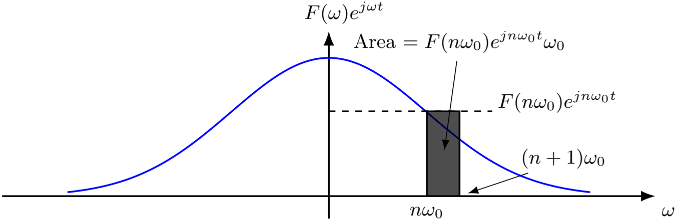
Aperiodic Signal Representation by Fourier Integral

- By replacing ω_0 with $\Delta\omega_0$ and since $T_0\infty$, $\Delta\omega \to 0$ and $\tilde{f}(t) \to f(t)$. Therefore

$$f(t) = \lim_{T_0\to\infty} \tilde{f}(t) = \lim_{\Delta\omega\to 0} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\Delta\omega) e^{jn\Delta\omega t} \Delta\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

- Each term in the summation on the right hand side of the above equation is the area of a rectangle of height $F(k\omega_0)e^{jn\omega_0 t}$ and width ω



Aperiodic Signal Representation by Fourier Integral

- We can $F(\omega)$ the **direct** Fourier transform of $f(t)$, and $f(t)$ the **inverse** Fourier transform of $F(\omega)$. We call $f(t)$ and $F(\omega)$ a Fourier transform pair. Symbolically, this statement is expressed as

$$F(\omega) = \mathcal{F}\left[f(t)\right] \quad \text{and} \quad f(t) = \mathcal{F}^{-1}\left[F(\omega)\right]$$

or

$$f(t) \Leftrightarrow F(\omega)$$

and

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Find the Fourier transform of $e^{-at}u(t)$.
Solution: By definition,

$$F(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{-1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty}$$

But $|e^{-j\omega t}| = 1$. Therefore, as $t \to \infty$, $e^{-(a+j\omega)t} = e^{-at}e^{-j\omega t} = 0$ if $a > 0$. Therefore

$$F(\omega) = \frac{1}{a+j\omega}, \qquad a > 0$$

Expressing $a + j\omega$ in the polar form as $\sqrt{a^2+\omega^2}e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}$, we obtain

$$F(\omega) = \frac{1}{\sqrt{a^2+\omega^2}}e^{-j\tan^{-1}\left(\frac{\omega}{a}\right)}$$

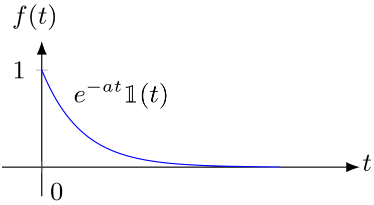
Therefore

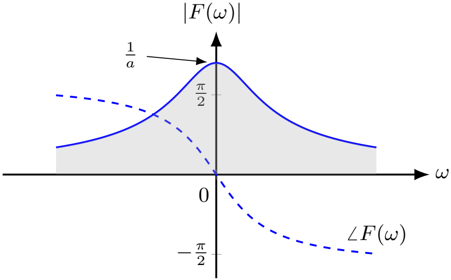
$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \text{and} \qquad \angle F(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Aperiodic Signal Representation by Fourier Integral

Example I cont.

The amplitude spectrum $|F(\omega)|$ and the phase spectrum $\angle F(\omega)$ are depicted as follow:





- $|F(\omega)|$ is an even function of ω
- $\angle F(\omega)$ is an odd function of ω

Existence of the Fourier Transform

- The existence of the Fourier transform is assured for any $f(t)$ satisfying the Dirichlet condition. The condition is

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

Because $|e^{-j\omega t}| = 1$ we obtain

$$|F(\omega)| \leq \int_{-\infty}^{\infty} |f(t)| dt$$

- However this is just a sufficient condition.
- The Fourier Transform is linear since

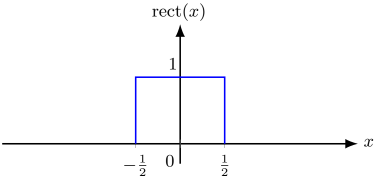
$$f_1(t) \Longleftrightarrow F_1(\omega) \qquad \text{and} \qquad f_2(t) \Longleftrightarrow F_2(\omega)$$

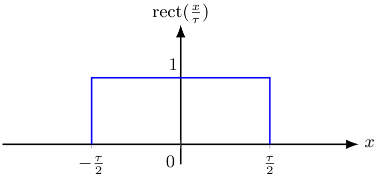
then

$$a_1 f_1(t) + a_2 f_2(t) \Longleftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$$

Unit Gate Function:
We define a unit gate function $\text{rect}(x)$ as a gate pulse of unit height and unit width, centered at the origin,

$$\text{rect}(x) = \begin{cases} 0, & |x| > \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 1, & |x| < \frac{1}{2} \end{cases}$$

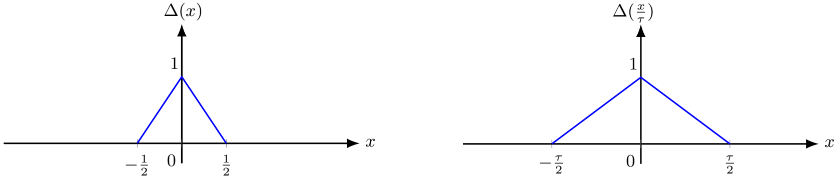




Unit Triangle Function:

We define a unit triangle function $\Delta(x)$ as a triangular pulse of unit height and unit width, centered at the origin,

$$\Delta(x) = \begin{cases} 0, & |x| > \frac{1}{2} \\ 1 - 2|x|, & |x| < \frac{1}{2} \end{cases}$$



Some Useful Functions

Interpolation Function $\text{sinc}(x)$

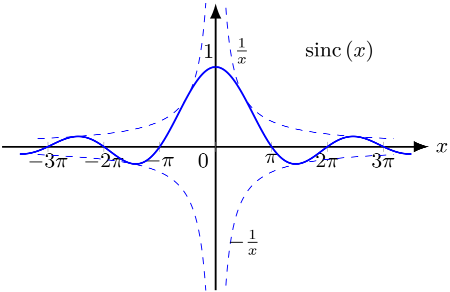
The function $\sin x/x$ is the "sine over argument" function denoted by $\text{sinc}(x)$. This function is very important in signal processing. It is also known as the **filtering or interpolation function**. We define

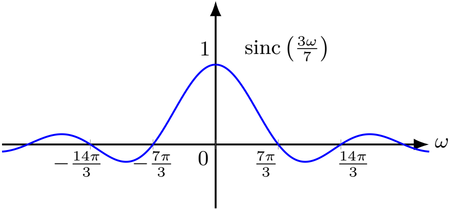
$$\text{sinc}(x) = \frac{\sin x}{x}$$

- $\text{sinc}(x)$ is an even function of x
- $\text{sinc}(x) = 0$ when $\sin x = 0$ except at $x = 0$, where it appears indeterminate. This means that $\text{sinc}(x) = 0$ for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
- Using L'Hôpital's rule, we find $\text{sinc}\, 0 = 1$.
- $\text{sinc}(x)$ is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $1/x$
- Consider $\text{sinc}(3\omega/7)$. Argument $\frac{3\omega}{7} = \pi$ when $\omega = 7\pi/\omega$. Therefore, the first zero of this function occurs at $\omega = 7\pi/3$.

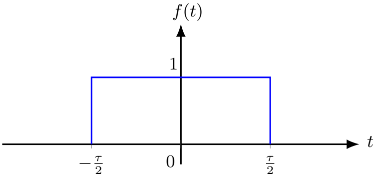
Some Useful Functions

Interpolation Function sinc(x)





Find the Fourier transform of $f(t) = \text{rect}\ \frac{t}{\tau}$



Solution:

$$F(\omega) = \int_{-\infty}^{\infty} \text{rect}\ \frac{t}{\tau} e^{-j\omega t} dt$$

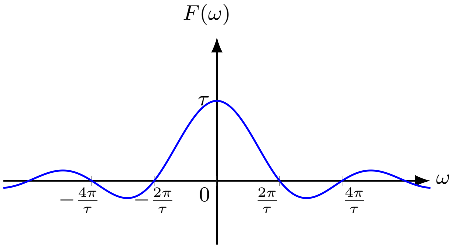
Since $\text{rect}\ \frac{t}{\tau} = 1$ for $|t| < \frac{\tau}{2}$ and since it is zero for $|t| > \frac{\tau}{2}$,

$$F(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

Some Useful Functions

Example II

$$
\begin{aligned}
F(\omega) &= \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt \\
&= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} \\
&= \tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)
\end{aligned}
$$



Fourier Transform

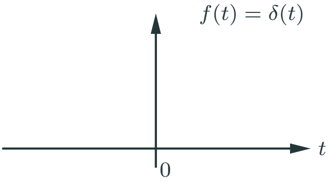
unit impulse

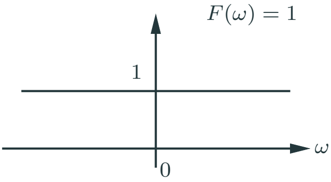
Find the Fourier transform of the unit impulse $\delta(t)$.

$$\mathcal{F}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

or

$$\delta(t) \Longleftrightarrow 1$$





Fourier Transform

unit impulse

Fine the inverse Fourier transform of $\delta(\omega)$

$$\mathcal{F}^{-1}\left[\delta(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}$$

Therefore

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega)$$

$$1 \Longleftrightarrow 2\pi\delta(\omega)$$

Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.
Solution:
Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}\left[\delta(\omega - \omega_0)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Therefore

$$\begin{aligned} \frac{1}{2\pi} e^{j\omega_0 t} &\Longleftrightarrow \delta(\omega - \omega_0) \\ e^{j\omega_0 t} &\Longleftrightarrow 2\pi\delta(\omega - \omega_0) \end{aligned}$$

- To represent the everlasting exponential $e^{j\omega_0 t}$, we need a single everlasting exponential $e^{j\omega t}$ with $\omega = \omega_0$.
- We also have

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0)$$

Find the Fourier transforms of the everlasting sinusoid $\cos\omega_0 t$

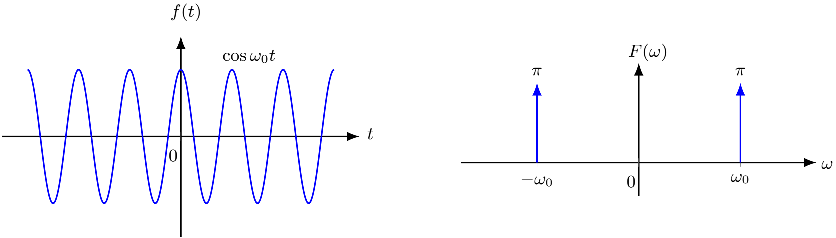
Solution:

Recall the Euler formula

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

Using the previous result, we have

$$\cos\omega_0 t \Longleftrightarrow \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$



Find the Fourier transforms of the everlasting sinusoid $\cos \omega_0 t$

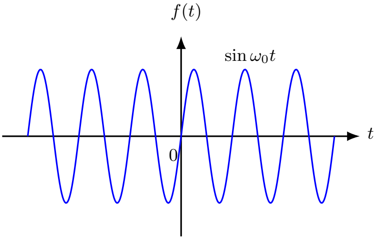
Solution:

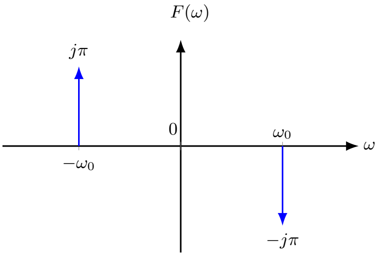
Recall the Euler formula

$$\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

Using the previous result, we have

$$\sin \omega_0 t \Longleftrightarrow j\pi \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$$





Fourier Transform

Unit step

Find the Fourier transform of the unit step function $u(t)$.
Solution: Direct integration

$$U(\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-j\omega t}dt = \left.\frac{-1}{j\omega}e^{-j\omega t}\right|_{0}^{\infty}$$

The upper limit of $e^{-j\omega t}$ as $t \to \infty$ yields an indeterminate answer. ($e^{-j\omega\infty}$) We can consider $u(t)$ as a decaying exponential $e^{-at}u(t)$ in the limit as $a \to 0$. Thus

$$u(t) = \lim_{a\to 0} e^{-at}u(t)$$

and

$$U(\omega) = \lim_{a\to 0} \mathcal{F}\left\{e^{-at}u(t)\right\} = \lim_{a\to 0} \frac{1}{a + j\omega}$$

Separate real and imaginary part yields

$$U(\omega) = \lim_{a\to 0}\left[\frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2}\right] = \lim_{a\to 0}\left[\frac{a}{a^2+\omega^2}\right] + \frac{1}{j\omega}$$

The function $a/(a^2+\omega^2)$ has interesting properties. First, the area under this function is π regardless of the value of a

$$\int_{-\infty}^{\infty} \frac{a}{a^2+\omega^2} d\omega = \tan^{-1}\frac{\omega}{a}\Big|_{-\infty}^{\infty} = \pi$$

Second, when $a \to 0$, this function approaches zero for all $\omega \neq 0$, and all its area (π) is concentrated at a single point $\omega = 0$. Clearly, as $a \to 0$, this function approaches an impulse of strength π. Thus

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

Reference

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