Fourier Series

An Introduction

Sudchai Boonto

Department of Control Systems and Instrumentation Engineering

Fourier Sine Series

Starting with $\sin(t)$

- It has period 2π since $\sin(t+2\pi) = \sin(t)$,
- It is an odd function, since $\sin(-t) = -\sin(t)$,
- and $\sin(t) = 0$ when $t = 0$ and $t = \pi$.

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- and $\sin(t) = 0$ when $t = 0$ and $t = \pi$.

The combinations of the sines is also have above three properties

Sine series

Fourier sine series is defined by:

$$S(t) = b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(3t) + \cdots = \sum_{n=1}^{\infty} b_n \sin(nt)$$

If the numbers $b_1, b_2, b_3, \ldots$ drop off quickly enough (decay rate) then the sum $S(t)$ will inherit all three properties:

Periodic $S(t+2\pi) = S(t)$ **Odd** $S(-t) = -S(t)$ $S(0) = S(\pi) = 0$

Fourier Sine series

Fourier (Mathematicians in France) suggests that any odd periodic function $S(t)$ could be expressed as an infinite series of sines. The problem is

$$f(t) \approx \sum_{n=1}^{\infty} b_n \sin(nt),$$

where $f(t)$ is an odd function.

Problem!

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$$f(t) \approx \sum_{n=1}^{\infty} b_n \sin(nt),$$

where $f(t)$ is an odd function.

Problem!

- What are the value of b_k that multiplies $\sin(kt)$?
- Suppose $S(x) = \sum b_n \sin(nt)$. Multiply both sides by $\sin(kt)$ and integrate from 0 to π:

$$\int_0^{\pi} S(t) \sin(kt) dt = \int_0^{\pi} b_1 \sin(t) \sin(kt) dt + \cdots + \int_0^{\pi} b_k \sin(kt) \sin(kt) dt + \cdots$$

- On the right hand side, all integrals are zero except the red on with $n = k$.

Orthogonal

To see why the red term of the previous slide is zero.

Definition

The sine are orthogonal if

$$\int_0^\pi \sin(nt)\sin(kt)dt = 0 \text{ if } n \neq k$$

Proof: Since

$$\int_0^\pi \cos(mt)dt = \left[\frac{\sin mt}{m}\right]_0^\pi = 0 - 0$$

Then

$$\sin(nt)\sin(kt) = \frac{1}{2}\cos((n-k)t) - \frac{1}{2}\cos((n+k)t) \Rightarrow \int_0^\pi \sin(t)\sin(kt)dt = 0$$

Orthogonal

If $n = k$

$$\int_0^\pi \sin(kt)\sin(kt)dt = \int_0^\pi \sin^2(kt)dt = \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos(2kt)\right) dt = \frac{\pi}{2}$$

Then, we have

$$\int_0^\pi S(t)\sin(kt)dt = \int_0^\pi b_k \sin(kt)\sin(kt)dt = b_k \frac{\pi}{2}$$
$$b_k = \frac{2}{\pi}\int_0^\pi S(t)\sin(kt)dt = \frac{1}{\pi}\int_{-\pi}^\pi S(t)\sin(kt)dt$$

Note: Since both $S(t)$ and $\sin(kt)$ are odd function, the multiplication of two odd function is even function. Then integrals from $-\pi$ to 0 and from 0 to π are equal.

Sine coefficients $S(-t) = -S(t)$

$$b_k = \frac{2}{\pi}\int_0^\pi S(t)\sin(kt)dt = \frac{1}{\pi}\int_{-\pi}^\pi S(t)\sin(kt)dt$$

Fourier Cosine Series

Cosine series

$$C(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt)$$

The sum $C(t)$ has two properties:

Periodic $C(t + 2\pi) = C(t)$ **Even** $C(t) = C(-t)$

Problem: What are the value of a_0 and a_k?

Fourier Cosine Series

Cosine series

$$C(t) = a_0 + a_1 \cos(t) + a_2 \cos(2t) + \cdots = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt)$$

The sum $C(t)$ has two properties:

Periodic $C(t + 2\pi) = C(t)$ **Even** $C(t) = C(-t)$

Problem: What are the value of a_0 and a_k?

The constant term a_0 is the average value of the function $C(t)$. We use the fact that $\int_0^\pi \cos(nt)dt = 0$. Then

$$\int_0^\pi C(t)dt = \int_0^\pi a_0 dt + 0 \Rightarrow a_0 = \frac{1}{\pi}\int_0^\pi C(t)dt = \frac{1}{2\pi}\int_{-\pi}^\pi C(t)dt$$

The other cosine coefficients a_k come from the *orthogonality of cosines*.

- As with sines, we multiply both sides of $C(t)$ by $\cos(kt)$ and integrate from 0 to π:

$$\int_0^\pi C(t)\cos(kt)dt = \int_0^\pi a_0 \cos(kt)dt + \int_0^\pi a_1 \cos(t)\cos(kt)dt + \cdots$$
$$+ \int_0^\pi a_k \cos^2(kt)dt + \cdots .$$

- On the right side, only the red term can be nonzero. (proof that the other terms are zero by yourself)
- We have

$$\int_0^\pi \cos^2(kt)dt = \int_0^\pi \left(\frac{1}{2} + \frac{1}{2}\cos(2kt)\right) dt = \frac{\pi}{2}$$

Then, we have

$$\int_0^\pi C(t)\cos(kt)dt = \int_0^\pi a_k \cos(kt)\cos(kt)dt = a_k \frac{\pi}{2} \Rightarrow a_k = \frac{2}{\pi}\int_0^\pi C(t)\cos(kt)dt$$

Cosine coefficients $C(-t) = C(t)$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{\pi} C(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(t) dt \\
a_k &= \frac{2}{\pi} \int_0^{\pi} C(t) \cos(kt) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} C(t) \cos(kt) dt
\end{aligned}
$$

Complete Fourier Series

Since the half-period $[0,\ \pi]$, the sines are not orthogonal to all the cosines. ($\int_0^\pi \sin(t)dt$ is not zero.) So for functions $F(t)$ that are not *odd or even*, we must move to the complete series (sines plus cosines) on the full interval.

Complete Fourier series

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)dt, \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos(kt)dt,$$

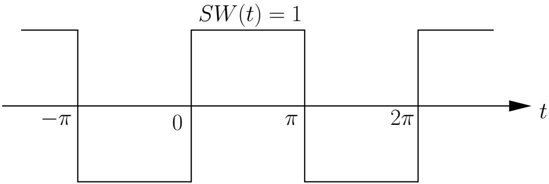
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin(kt)dt$$

Note:

$$C(t) = F_{\text{even}}(t) = \frac{F(t)+F(-t)}{2}, \qquad S(t) = F_{\text{odd}} = \frac{F(t)-F(-t)}{2}$$

Example: odd function

Find the Fourier sine coefficients b_k of the odd square wave $SW(t)$.



Solution: We have no cosine terms, since $SW(t)$ is an odd function (You could proof it that all a_i terms are zeros). For $k = 1, 2, \ldots$ Since $S(t) = 1$ between 0 and π.

$$b_k = \frac{2}{\pi}\int_0^\pi S(t)\sin(kt)dt = \frac{2}{\pi}\int_0^\pi \sin(kt)dt = \frac{2}{\pi}\left[\frac{-\cos(kt)}{k}\right]_0^\pi$$

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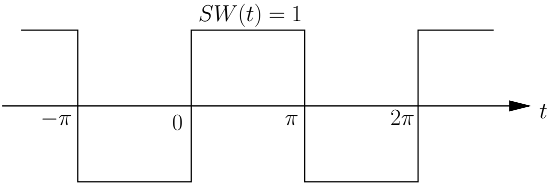

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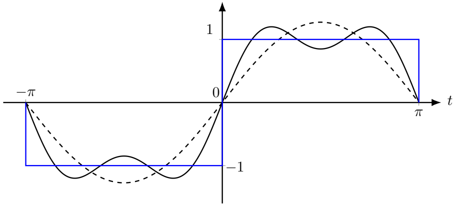
- $k = 1,\ \frac{4}{\pi}$
- $k = 2,\ 0$
- $k = 3,\ \frac{4}{3\pi}$
- $k = 4,\ 0$
- $k = 5,\ \frac{4}{5\pi}$
- $k = 6,\ 0$
- $k = 7,\ \frac{4}{7\pi}$
- $k = 8,\ 0$
- in general

$$b_k = \begin{cases} \frac{2}{\pi}\frac{2}{k}, & k = \text{ odd} \\ 0, & k = \text{ even} \end{cases}$$

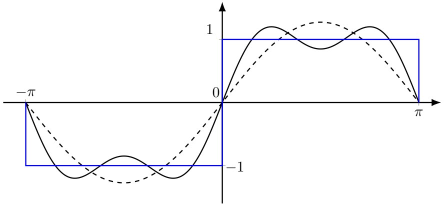
Example: odd function

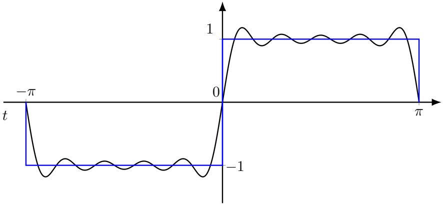


Dashed curve: $\dfrac{4}{\pi}\dfrac{\sin(t)}{1}$

Solid curve: $\dfrac{4}{\pi}\left(\dfrac{\sin(t)}{1}+\dfrac{\sin(3t)}{3}\right)$

Example: odd function





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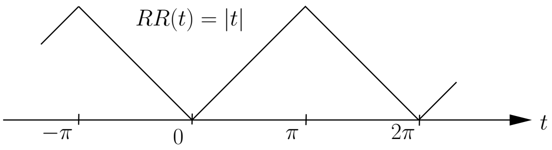
Solid curve: $\frac{4}{\pi}\left(\frac{\sin(t)}{1}+\frac{\sin(3t)}{3}\right)$

Solid curve:
$\frac{4}{\pi}\left(\frac{\sin(t)}{1}+\cdots+\frac{\sin(9t)}{9}\right)$

Example: ramp function



Find the cosine coefficients of the even ramp $RR(t)$ function.

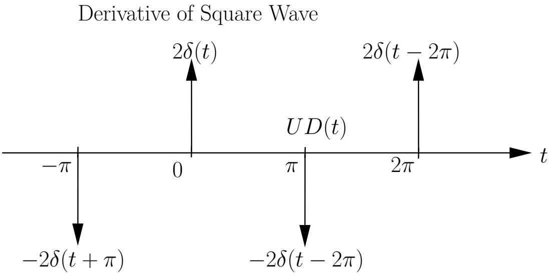
Solution: You can see that b_k terms are zero because the $RR(t)$ is an even function. Instead of direct find the Fourier coefficient of the series, we can integrate the square wave series $SW(t)$ and add a_0. The average ramp height is $a_0 = \pi/2$. Since the sine series of the $SW(t)$ is:

$$SW(t) = \frac{4}{\pi}\left[\frac{\sin(t)}{1} + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \frac{\sin(7t)}{7} + \cdots\right]$$

Then,

$$RR(t) = \frac{\pi}{2} - \frac{\pi}{4}\left[\frac{\cos(t)}{1^2} + \frac{\cos(3t)}{3^2} + \frac{\cos(5t}{5^2} + \frac{\cos(7t)}{7^2} + \cdots\right]$$

Example: Up-down function



Since the $UP(t)$ is the derivative of the square wave function. Then take the derivative of every term to produce cosines in the up-down delta function:

$$UD(t) = \frac{4}{\pi}\left[\cos(t) + \cos(3t) + \cos(5t) + \cos(7t) + \cdots\right]$$

The Fourier Series for a Delta Function

Find the (cosine) coefficients of the *delta function* $\delta(t)$, made 2π-periodic. The impulse $\delta(t)$ occurs at $t = 0$ and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \text{with } 2\pi - \text{periodic} \qquad \int_{-\pi}^{\pi} \delta(t)dt = 1$$

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$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \text{with } 2\pi - \text{periodic} \qquad \int_{-\pi}^{\pi} \delta(t)dt = 1$$

Imagine that the $\delta(t)$ is an even function. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(t)dt = \frac{1}{2\pi} \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(t)\cos(kt)dt = \frac{1}{\pi}, \forall k$$

Then the series for the delta function has all cosines in equal amounts: **No decay**.

$$\begin{aligned}\delta(t) &= \frac{1}{2\pi} + \frac{1}{\pi}\left[\cos(t) + \cos(2t) + \cos(3t) + \cdots\right] \\ &= \frac{1}{2\pi}\left[1 + 2\cos(t) + 2\cos(2t) + 2\cos(3t) + \cdots\right]\end{aligned}$$

The Fourier Series for a Delta Function

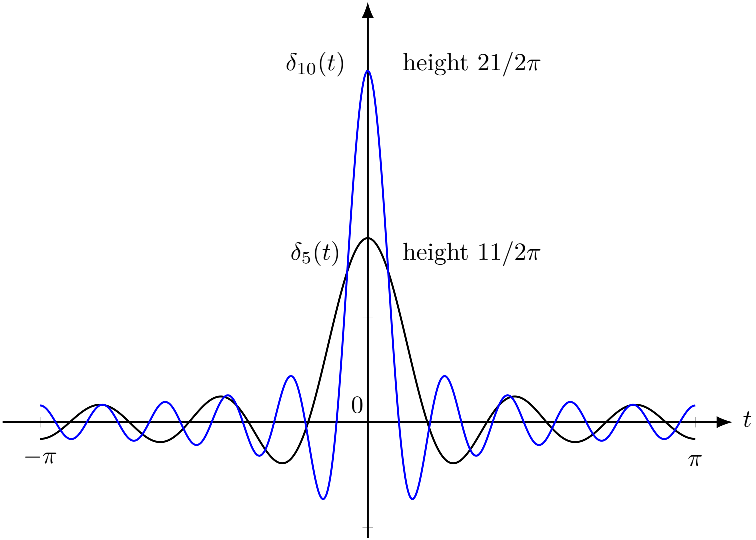
Since $2\cos(t) = e^{jt} + e^{-jt}$, we have

$$\begin{aligned}
\delta_N &= \frac{1}{2\pi}\left[1 + e^{jt} + e^{-jt} + \cdots + e^{jNt} + e^{-jNt}\right] \\
&= \frac{1}{2\pi}\left[1 + e^{jt} + e^{j2t} + \cdots + e^{jNt} + e^{-jt} + e^{-j2t} + \cdots + e^{-jNt}\right] \\
&= \frac{1}{2\pi}\left[\frac{1 - e^{j(N+1)x}}{1 - e^{jx}} + \frac{-1 + e^{-jNx}}{1 - e^{jx}}\right] \\
&= \frac{1}{2\pi}\frac{e^{j(N+1)x} + e^{-jNx}}{1 - e^{jx}} = \frac{e^{j\frac{1}{2}x}\left(e^{jNx}e^{j\frac{1}{2}x} + e^{-jNx}e{-j\frac{1}{2}x}\right)}{e^{j\frac{1}{2}x}\left(e^{-j\frac{1}{2}x} + e^{j\frac{1}{2}x}\right)} \\
&= \frac{1}{2\pi}\frac{\sin(N + \frac{1}{2})t}{\sin(\frac{1}{2}t)}
\end{aligned}$$

Note: we use the fact that

$$\sum_{k=0}^{N} a^k = \frac{1 - a^{N+1}}{1 - a}$$

The Fourier Series for a Delta Function



Example

Find the a_k and b_k if

$$F(t) = \begin{cases} \frac{1}{h}, & 0 < t < h \\ 0, & h < t < 2\pi \end{cases}$$

Solution: The integrals for a_0 and a_k and b_k stop at $t = h$ where $F(t)$ drops to zero. The coefficients decay like $1/k$ because of the jump at $x = 0$ and the drop at $x = h$:

$$a_0 = \frac{1}{2\pi} \int_0^h \frac{1}{h} dt = \frac{1}{2\pi} = \text{ average}$$

$$a_k = \frac{1}{\pi h} \int_0^h \cos(kt) dt = \frac{\sin(kh)}{\pi kh}$$

$$b_k = \frac{1}{\pi h} \int_0^h \sin(kt) dt = \frac{1 - \cos(kh)}{\pi kh}$$

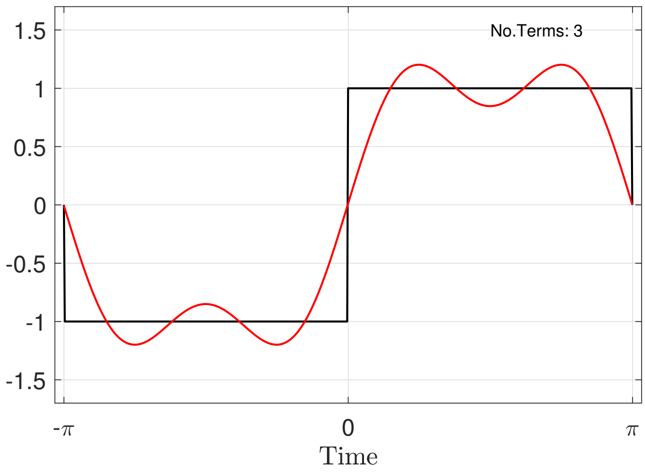
Matlab code

```
% fourier plot
close all; clear all;
dx = 0.01; L = 2*pi;
x = 0:dx:L;
xp = 0:dx:pi-dx;

% one fourth of data point;
n = length(x);
npart = length(xp);

% define function
f = zeros(size(x));
f(1) = 0; f(2:npart) = -1;
f(npart+1:2*npart) = 1;
f(2*npart+1) = 0;
```

Matlab code

```
for N = 1:3,             % three terms
    hold off;
    % plot original function
    plot(x, f, 'k', 'linewidth', 1.2)
    grid;

    % determine sine and cosine terms and sum them all
    A0 = sum(f.*ones(size(x))) * dx * 2/L;
    fFS = A0/2;
    for k = 1:N
        Ak = sum(f.*cos(2*pi*k*x/L))*dx*2/L;
        Bk = sum(f.*sin(2*pi*k*x/L))*dx*2/L;
        fFS = fFS + Ak*cos(2*k*pi*x/L) ...
            + Bk*sin(2*k*pi*x/L);
    end
    hold on;
    plot(x,fFS, 'r-', 'linewidth', 1.2);
    axis([-0.1,2*pi+0.1,-1.7, 1.7]);
end
```

Matlab code



Complex Exponentials $c_k e^{-jkt}$

We can combine a_k and b_k into one c_k.

Complex Fourier series

$$F(t) = c_0 + c_1 e^{jt} + c_{-1} e^{-jt} + \cdots = \sum_{n=-\infty}^{\infty} c_n e^{jnt}$$

- Every $c_n = c_{-n}$, we can combine e^{jnt} with e^{-jnt} into $2\cos(nt)$. Then the summation above is the cosine series for an even function.
- If every $c_n = -c_{-n}$, we use $e^{jnt} = -e^{-jnt} = 2j\sin(nt)$, then the summation is the sine series for an odd function and the c_k are pure imaginary.
- To find c_k, we start with multiplication of $F(t)$ and e^{-jkt} and integrate from $-\pi$ to π :

$$\int_{-\pi}^{\pi} F(t) e^{-jkt} dt = \int_{-\pi}^{\pi} c_0 e^{-jkt} dt + \int_{-\pi}^{\pi} c_1 e^{jt} e^{-jkt} dt + \cdots$$
$$+ \int_{-\pi}^{\pi} c_k e^{jkt} e^{-jkt} dt + \cdots$$

Complex Exponentials $c_k e^{-jkt}$

Every integral on the right hand side is zero:

$$\int_{-\pi}^{\pi} c_0 e^{-jt} dt = \int_{-\pi}^{\pi} c_0 \left(\cos(kt) - j\sin(kt)\right) dt = 0, \qquad \omega = k \ \Rightarrow 2\pi = kT$$

$$\int_{-\pi}^{\pi} c_1 e^{jt} e^{-jkt} dt = \int_{-\pi}^{\pi} c_1 e^{-j(k-1)t} dt = \int_{-\pi}^{\pi} c_1 \left[\cos((k-1)t) - j\sin((k-1)t)\right] dt = 0,$$

$$\omega = k - 1 \ \Rightarrow 2\pi = (k-1)T$$

Complex Exponentials $c_k e^{-jkt}$

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$$\omega = k-1 \;\Rightarrow 2\pi = (k-1)T$$

The red term is

$$\int_{-\pi}^{\pi} c_k e^{jkt} e^{-jkt} dt = \int_{-\pi}^{\pi} c_k dt = 2\pi c_k$$

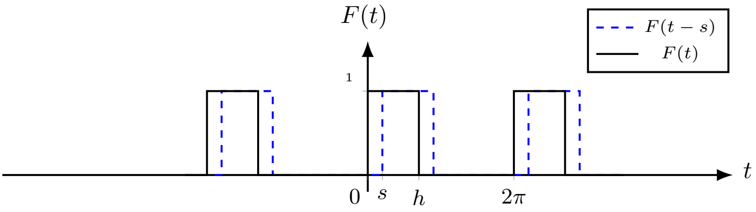
$$\int_{-\pi}^{\pi} F(t) e^{-jkt} dt = 2\pi c_k, \text{ for } k = 0, \pm 1, \ldots, l$$

Then,

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) e^{-jkt} dt \text{ for } k = 0, \pm 1, \ldots, l$$

$c_0 = a_0$ the every of $F(t)$

Example



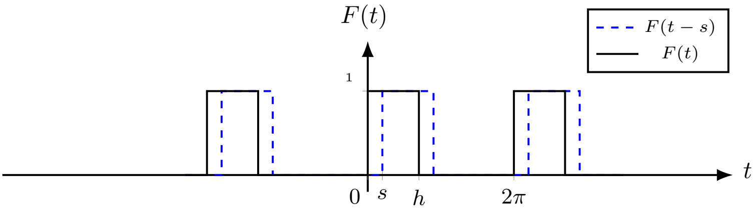
Find c_k for the 2π-periodic shifted box

$$F(t) = \begin{cases} 1 & , s \le t \le s+h \\ 0 & , \text{ elsewhere in } [-\pi, \pi] \end{cases}$$

Solution: We have $(T = 2\pi, \omega = 1)$

$$c_k = \frac{1}{2\pi} \int_s^{s+h} 1 e^{-jkt} dt = \frac{1}{2\pi} \left[\frac{e^{-jkt}}{-jk} \right]_s^{s+h} = e^{-jks} \left(\frac{1 - e^{-jkh}}{2\pi jk} \right)$$

Example



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Note: Actually, shift $F(t)$ to $F(t-s) \leftrightarrow$ Multiply every c_k by e^{-jks}.

Example

From the previous example, if we shift $F(t)$ to the left by $s = h/2$, the pulse becomes symmetry around $t = 0$. This even function $F_c(t)$ equals 1 on the interval from $-h/2$ to $h/2$. We don't need to re-calculate the c_k.

$$s = -\frac{h}{2} \qquad c_k = e^{-jk(-\frac{h}{2})}\left(\frac{1 - e^{-jkh}}{2\pi jk}\right) = \frac{1}{2\pi}\frac{\sin(kh/2)}{k/2}$$

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$$c_0 = \frac{1}{2\pi}\int_{-h/2}^{h/2} 1dt = \frac{1}{2\pi}[t]_{-h/2}^{h/2} = \frac{h}{2\pi}$$

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$$c_0 = \frac{1}{2\pi}\int_{-h/2}^{h/2} 1dt = \frac{1}{2\pi}\left[t\right]_{-h/2}^{h/2} = \frac{h}{2\pi}$$

Then

$$\frac{F_c(t)}{h} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{\sin(kh/2)}{kh/2}e^{jkt} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{kh}{2}\right)e^{jkt}$$

Reference

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