Lecture 8: Discrete-Time System Analysis Using the Z-Transform

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Outline

- z-Transform
- Inverse z-Transform
- Some properties of the z-Transform
- z-Transform Solution of Linear Difference Equations

z-Transform

The z-transform is defined by

$$F[z] = \sum_{k=-\infty}^{\infty} f[k] z^{-k}$$
$$f[k] = \frac{1}{2\pi j} \oint F[z] z^{k-1} dz$$

We are restricted only to the analysis of causal systems with causal input. In **the unilateral z-transform**, the signals are restricted to be causal; that is, they start at $k = 0$. The unilateral z-transform is defined by

$$F[z] = \sum_{k=0}^{\infty} f[k] z^{-k},$$

where z is complex in general.

Find the z-transform of a signal $\gamma^k u[k]$.

$$F[z] = \sum_{k=0}^{\infty} \gamma^k u[k] z^{-k} = \sum_{k=0}^{\infty} \gamma^k z^{-k}$$
$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

From the well-known geometric progression and its sum:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ if } |x| < 1$$

we have

$$F[z] = \frac{1}{1-\frac{\gamma}{z}}, \quad \left|\frac{\gamma}{z}\right| < 1$$
$$= \frac{z}{z-\gamma}, \quad |z| > |\gamma|$$

$F[z]$ exists only for $|z| > |\gamma|$. This region of $|z|$ is called the **region of convergence**.

Find the z-transforms of (a) $\delta[k]$, (b) $u[k]$, (c) $\cos \beta k u[k]$
By definition

$$
\begin{aligned}
F[z] &= \sum_{k=0}^{\infty} f[k] z^{-k} \\
&= f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \frac{f[3]}{z^3} + \cdots
\end{aligned}
$$

(a) For $f[k] = \delta[k]$, $f[0] = 1$, and $f[2] = f[3] = f[4] = \cdots = 0$. Therefore

$$\delta[k] \overset{\mathcal{Z}}{\longleftrightarrow} 1 \quad \text{for all } z$$

(b) For $f[k] = u[k], f[0] = f[1] = f[3] = \cdots = 1$. Therefore

$$
\begin{aligned}
F[z] &= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1 \\
&= \frac{z}{z-1} \qquad |z| > 1
\end{aligned}
$$

Therefore

$$u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-1} \qquad |z| > 1.$$

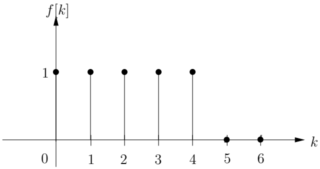
(c) Recall that $\cos \beta k = \left(e^{j\beta k} + e^{-j\beta k}\right)/2$. Moreover,

$$e^{\pm j\beta k} u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z - e^{\pm j\beta}} \qquad |z| > |e^{\pm j\beta}| = 1$$

Therefore

$$\begin{aligned} F[z] &= \frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right] \\ &= \frac{z(z-\cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1 \end{aligned}$$

Find the z-transforms of a signal shown in Figure below



Here $f[0] = f[1] = f[2] = f[3] = f[4] = 1$ and $f[5] = f[6] = \cdots = 0$. Therefore,

$$F[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$
$$= \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

or

$$F[z] = \frac{\left(\frac{1}{z}\right)^5 - \left(\frac{1}{z}\right)^0}{\frac{1}{z} - 1} = \frac{z}{z-1}(1 - z^{-5})$$

Inverse z-Transform

Many of the transforms $F[z]$ of practical interest are rational functions. Such functions can be expressed as a sum of simpler functions using partial fraction expansion.

Find the inverse z-transform of

$$F[z] = \frac{8z - 19}{(z-2)(z-3)}$$

Expanding $F[z]$ into partial fractions yields

$$F[z] = \frac{8z - 19}{(z-2)(z-3)} = \frac{3}{z-2} + \frac{5}{z-3}$$

From z-transform Table Pair 6, we have

$$f[k] = \left[3(2)^{k-1} + 5(3)^{k-1}\right] u[k-1]$$

This result is not convenient. We prefer the form that in multiplied by $u[k]$ rather than $u[k-1]$. We will expand $F[z]/z$ instead of $F[z]$. For this case

$$\frac{F[z]}{z} = \frac{8z - 19}{z(z-2)(z-3)} = \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$$

Multiplying both sides by z yields

$$F[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

From Pairs 1 and 7 in the z-transform table, it follows that

$$f[k] = -\frac{19}{6}\delta[k] + \left[\frac{3}{2}(2)^k + \frac{5}{3}(3)^k\right]u[k]$$

Find the inverse z-transform of

$$F[z] = \frac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$$

and

$$\begin{aligned}\frac{F[z]}{z} &= \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} \\ &= \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}\end{aligned}$$

Using a cover up method yields

$$\begin{aligned}k &= \left.\frac{2z^2 - 11z + 12}{(z-2)^3}\right|_{z=1} = -3 \\ a_0 &= \left.\frac{2z^2 - 11z + 12}{(z-1)}\right|_{z=2} = -2\end{aligned}$$

Therefore

$$\frac{F[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

By using short cuts method, we multiply both sides of the equation by z and let $z \to \infty$. This yields

$$0 = -3 - 0 + 0 + a_2 \Longrightarrow a_2 = 3$$

Another unknown a_1 is readily determined by letting z take any convenient value, $z = 0$, on both sides. This step yields

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2} \Longrightarrow a_1 = -1$$

Therefore

$$\frac{F[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

$$F[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

Now the use of Table, Pairs 7 and 10 yields

$$\begin{aligned} f[k] &= \left[-3 - 2\frac{k(k-1)}{8}(2)^k - \frac{k}{2}(2)^k + 3(2)^k\right] u[k] \\ &= -\left[3 + \frac{1}{4}(k^2 + k - 12)2^k\right] u[k] \end{aligned}$$

Complex Poles

$$F[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$$

Method of First-Order Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

We find the partial fraction of $F[z]/z$ using the "cover up" method:

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

and

$$F[z] = 2\frac{z}{z-1} + (1.6e^{-j2.246})\frac{z}{z-3-j4} + (1.6e^{j2.246})\frac{z}{z-3+j4}$$

The inverse transform of the first time on the right-hand side is $2u[k]$. The inverse transform of the remeaining two terms can be obtained from z-Transform Table Pair 12b by identifying $\frac{r}{2} = 1.6$, $\theta = -2.246$ rad, $\gamma = 3 + j4 = 5e^{j0.927}$, so that $|\gamma| = 5$, $\beta = 0.927$. Therefore

$$f[k] = [2 + 3.2(5)^k \cos(0.927k - 2.246)]u[k]$$

Method of Quadratic Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

Multiplying both sides by z and letting $z \to \infty$, we find

$$0 = 2 + A \Longrightarrow A = -2$$

and

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

Inverse z-Transform

Examples

To find B we let z take any convenient value, say $z = 0$. This step yields

$$\frac{-34}{25} = -2 + \frac{B}{25}$$

Multiplying both sides by 25 yields

$$-34 = -50 + B \Longrightarrow B = 16$$

Therefore

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

and

$$F[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

Inverse z-Transform

Examples

We now usr z-Transform table Pair 12c where we identify $A = -2$, $B = 16$, $|\gamma| = 5$, $a = -3$. Therefore

$$r = \sqrt{\frac{100 + 256 - 192}{25 - 9}} = 3.2,\ \ \beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927 \text{ rad}$$

and

$$\theta = \tan^{-1}\left(\frac{-10}{-8}\right) = -2.246 \text{ rad.}$$

so that

$$f[k] = \Big[2 + 3.2(5)^k \cos(0.927k - 2.246)\Big]\, u[k]$$

Right Shift (Delay)

If

$$f[k]u[k] \Longleftrightarrow F[z]$$

then

$$f[k-1]u[k-1] \Longleftrightarrow \frac{1}{z}F[z]$$

and

$$f[k-m]u[k-m] \Longleftrightarrow \frac{1}{z^m}F[z]$$

and

$$f[k-1]u[k] \Longleftrightarrow \frac{1}{z}F[z] + f[-1]$$

Repeated application of this property yields

$$f[k-2]u[k] \Longleftrightarrow \frac{1}{z}\left[\frac{1}{z}F[z] + f[-1]\right] + f[-2]$$

Some properties of the z-Transform

Right Shift (Delay) cont.

$$f[k-2]u[k] = \frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2]$$

and

$$f[k-m]u[k] \Longleftrightarrow z^{-m}F[z] + z^{-m}\sum_{k=1}^{m} f[-k]z^{k}$$

Proof:

$$\mathcal{Z}\left\{f[k-m]u[k-m]\right\} = \sum_{k=0}^{\infty} f[k-m]u[k-m]z^{-k}$$

Recall that $f[k-m]u[k-m] = 0$ for $k < m$, so that the limits on the summation on the right-hand side can be taken form $k = m$ to ∞.

Some properties of the z-Transform

Right-Shifty (Delay) cont.

$$\mathcal{Z}\left\{f[k-m]u[k-m]\right\} = \sum_{k=m}^{\infty} f[k-m]z^{-k}$$

$$= \sum_{r=0}^{\infty} f[r]z^{-(r+m)} = \frac{1}{z^m}\sum_{r=0}^{\infty} f[r]z^{-r} = \frac{1}{z^m}F[z]$$

$$\mathcal{Z}\left\{f[k-m]u[k]\right\} = \sum_{k=0}^{\infty} f[k-m]z^{-k} = \sum_{r=-m}^{\infty} f[r]z^{-(r+m)}$$

$$= z^{-m}\left[\sum_{r=-m}^{-1} f[r]z^{-r} + \sum_{r=0}^{\infty} f[r]z^{-r}\right]$$

$$= z^{-m}\sum_{k=1}^{m} f[-k]z^{k} + z^{-m}F[z]$$

Some properties of the z-Transform

Left-Shifty (Advance)

If

$$f[k]u[k] \Longleftrightarrow F[z]$$

then

$$f[k+1]u[k] \Longleftrightarrow zF[z] - zf[0]$$

Repeated application of this property yields

$$\begin{aligned} f[k+2]u[k] \Longleftrightarrow & z\left\{z(F[z] - zf[0]) - f[1]\right\} \\ &= z^2F[z] - z^2f[0] - zf[1] \end{aligned}$$

and

$$f[k+m]u[k] \Longleftrightarrow z^mF[z] - z^m\sum_{k=0}^{m-1} f[k]z^{-k}$$

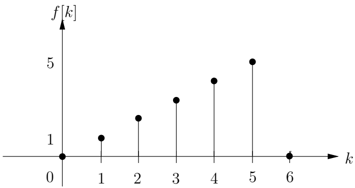
Proof: By definition

$$
\begin{aligned}
\mathcal{Z}\left\{f[k+m]u[k]\right\} &= \sum_{k=0}^{\infty} f[k+m]z^{-k} \\
&= \sum_{r=m}^{\infty} f[r]z^{-(r-m)} \\
&= z^m \sum_{r=m}^{\infty} f[r]z^{-r} \\
&= z^m \left[\sum_{r=0}^{\infty} f[r]z^{-r} - \sum_{r=0}^{m-1} f[r]z^{-r}\right] \\
&= z^m F[z] - z^m \sum_{r=0}^{m-1} f[r]z^{-r}
\end{aligned}
$$

Some properties of the z-Transform

Left-Shifty (Advance) cont.

Find the z-transform of the signal $f[k]$ depicted in a Figure below.



The signal can be expressed as a product of k and a gate pulse $u[k] - u[k-6]$. Therefore

$$\begin{aligned} f[k] &= ku[k] - ku[k-6] = ku[k] - (k-6+6)u[k-6] \\ &= ku[k] - (k-6)u[k-6] + 6u[k-6] \end{aligned}$$

Because $u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-1}$ and $ku[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{(z-1)^2}$,

$u[k-6] \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{1}{z^6}\dfrac{z}{z-1} = \dfrac{1}{z^5(z-1)}$, and $(k-6)u[k-6] \overset{\mathcal{Z}}{\longleftrightarrow} \dfrac{1}{z^6}\dfrac{z}{(z-1)^2} = \dfrac{1}{z^5(z-1)^2}$

Some properties of the z-Transform

Left-Shifty (Advance) cont.

Therefore

$$\begin{aligned} F[z] &= \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)} \\ &= \frac{z^6 - 6z + 5}{z^5(z-1)^2} \end{aligned}$$

Some properties of the z-Transform
Convolution

The time convolution property state that if

$$f_1[k] \overset{\mathcal{Z}}{\longleftrightarrow} F_1[z] \quad \text{and} \quad f_2[k] \overset{\mathcal{Z}}{\longleftrightarrow} F_2[z],$$

then (**time convolution**)

$$f_1[k] * f_2[k] \overset{\mathcal{Z}}{\longleftrightarrow} F_1[z]F_2[z]$$

Proof:

$$\begin{aligned}
\mathcal{Z}\left\{f_1[k] * f_2[k]\right\} &= \mathcal{Z}\left[\sum_{m=-\infty}^{\infty} f_1[m]f_2[k-m]\right] \\
&= \sum_{k=-\infty}^{\infty} z^{-k} \sum_{m=-\infty}^{\infty} f_1[m]f_2[k-m]
\end{aligned}$$

Some properties of the z-Transform

Convolution cont.

Interchanging the order of summation,

$$\begin{aligned}
\mathcal{Z}\left[f_1[k] * f_2[k]\right] &= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{k=-\infty}^{\infty} f_2[k-m] z^{-k} \\
&= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{r=-\infty}^{\infty} f_2[r] z^{-(r+m)} \\
&= \sum_{m=-\infty}^{\infty} f_1[m] z^{-m} \sum_{r=-\infty}^{\infty} f_2[r] z^{-r} \\
&= F_1[z] F_2[z]
\end{aligned}$$

If

$$f[k]u[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} F[z]$$

then

$$\gamma^k f[k]u[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} F\left[\frac{z}{\gamma}\right]$$

Proof:

$$\mathcal{Z}\left\{\gamma^k f[k]u[k]\right\} = \sum_{k=0}^{\infty} \gamma^k f[k] z^{-k} = \sum_{k=0}^{\infty} f[k]\left(\frac{z}{\gamma}\right)^{-k} = F\left[\frac{z}{\gamma}\right]$$

If

$$f[k]u[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} F[z]$$

then

$$kf[k]u[k] \stackrel{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}F[z]$$

Proof:

$$-z\frac{d}{dz}F[z] = -z\frac{d}{dz}\sum_{k=0}^{\infty} f[k]z^{-k} = -z\sum_{k=0}^{\infty} -kf[k]z^{-k-1}$$
$$= \sum_{k=0}^{\infty} kf[k]z^{-k} = \mathcal{Z}\left\{kf[k]u[k]\right\}$$

z-Transform Solution of Linear Difference Equations

- The time-shift (left- or right-shift) property has set the stage for solving linear difference equations with constant coefficients.
- As in the case of the Laplace transform with differential equations, the z-transform converts difference equations into algebraic equations which are readily solved to find the solution in the z-domain.
- Taking the inverse z-transform of the z-domain solution yields the desired time-domain solution.

Solve

$$y[k+2] - 5y[k+1] + 6y[k] = 3f[k+1] + 5f[k]$$

if the initial conditions are $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$, and the input $f[k] = (2)^{-k}u[k]$. Since the given initial conditions are not suitable for the forward form, we transform the equation to the delay form:

$$y[k] - 5y[k-1] + 6y[k-2] = 3f[k-1] + 5f[k-2].$$

Clearly that we consider the solution when $k \geq 0$ then $y[k-j]$ means $y[k-j]u[k]$. Now

$$\begin{aligned} y[k]u[k] &\overset{\mathcal{Z}}{\longleftrightarrow} Y[z] \\ y[k-1]u[k] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z}Y[z] + y[-1] = \frac{1}{z}Y[z] + \frac{11}{6} \\ y[k-2]u[k] &\overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^2}Y[z] + \frac{1}{z}y[-1] + y[-2] = \frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36} \end{aligned}$$

z-Transform Solution of Linear Difference Equations

Examples cont.

Also

$$f[k] = (2)^{-k}u[k] = (2^{-1})^k u[k] = (0.5)^k u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z}{z-0.5}$$

$$f[k-1]u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z}F[z] + f[-1] = \frac{1}{z}\frac{z}{z-0.5} + 0 = \frac{1}{z-0.5}$$

$$f[k-2]u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2] = \frac{1}{z^2}F[z] + 0 + 0 = \frac{1}{z(z-0.5)}$$

Note that for causal input $f[k]$,

$$f[-1] = f[-2] = \cdots = f[-n] = 0$$

Hence

$$f[k-r]u[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{z^r}F[z]$$

Taking the z-transform of the difference equation and substituting the above results, we obtain

$$Y[z] - 5\left[\frac{1}{z}Y[z] + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right] = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

or

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] - \left(3 - \frac{11}{z}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

and

$$\begin{aligned}\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] &= \left(3 - \frac{11}{z}\right) + \frac{3z+5}{z(z-0.5)} \\ &= \frac{3z^2 - 9.5z + 10.5}{z(z-0.5)}\end{aligned}$$

Multiplication of both sides by z^2 yields

$$(z^2 - 5z + 6)Y[z] = \frac{z(3z^2 - 9.5z + 10.5)}{(z - 0.5)}$$

so that

$$Y[z] = \frac{z(3z^2 - 9.5z + 10.5)}{(z - 0.5)(z^2 - 5z + 6)}$$

and

$$\begin{aligned} \frac{Y[z]}{z} &= \frac{3z^2 - 9.5z + 10.5}{(z - 0.5)(z - 2)(z - 3)} \\ &= \frac{(26/15)}{z - 0.5} - \frac{(7/3)}{z - 2} + \frac{(18/5)}{z - 3} \end{aligned}$$

Therefore

$$Y[z] = \frac{26}{15}\left(\frac{z}{z - 0.5}\right) - \frac{7}{3}\left(\frac{z}{z - 2}\right) + \frac{18}{5}\left(\frac{z}{z - 3}\right)$$

and

$$y[k] = \left[\frac{26}{15}(0.5)^k - \frac{7}{3}(2)^k + \frac{18}{5}(3)^k \right] u[k]$$

- From the previous example, we found the total solution of the difference equation.
- It is easy to separate the solution into zero-input and zero-state components.
- We have to separate the response into terms arising from the input and terms arising from initial conditions.

From the previous example:

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] - \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} = \underbrace{\frac{3}{z-0.5} + \frac{5}{z(z-0.5)}}_{\text{terms arising from input}}$$

Therefore

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] = \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} + \underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

Multiplying both sides by z^2 yields

$$(z^2 - 5z + 6)Y[z] = \underbrace{z(3z-11)}_{\text{initial condition terms}} + \underbrace{\frac{z(3z+5)}{z-0.5}}_{\text{input terms}}$$

and

$$Y[z] = \underbrace{\frac{z(3z-11)}{z^2-5z+6}}_{\text{zero-input response}} + \underbrace{\frac{z(3z+5)}{(z-0.5)(z^2-5z+6)}}_{\text{zero-state response}}$$

We expand both terms on the right-hand side into modified partial fractions to yield

$$Y[z] = \underbrace{\left[5\left(\frac{z}{z-2}\right) - 2\left(\frac{z}{z-3}\right)\right]}_{\text{zero-input}} + \underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero-state}}$$

and

$$\begin{aligned} y[k] &= \left[\underbrace{5(2)^k - 2(3)^k}_{\text{zero-input}} \underbrace{- \frac{22}{3}(2)^k + \frac{28}{5}(3)^k + \frac{26}{15}(0.5)^k}_{\text{zero-state}}\right] u[k] \\ &= \left[-\frac{7}{3}(2)^k + \frac{18}{5}(3)^k + \frac{26}{15}(0.5)^k\right] u[k] \end{aligned}$$

Reference

1. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998.