Lecture 7: Time-Domain Analysis of Discrete-Time Systems

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Outline

- Discrete-Time System Equations
- E operator
- Response of Linear Discrete-Time Systems
- Zero-Input and Zero-State Response
- Convolution Sum
- Graphical Procedure for the Convolution Sum
- Classical Solution of Linear Difference Equations

Discrete-Time System Equations

Advanced operator form:

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] =$$
$$b_m f[k+m] + b_{m-1} f[k+m-1] + \cdots + b_1 f[k+1] + b_0 f[k]$$

- The left-hand side of this form consists of the output at instants $k+n$, $k+n-1$, $k+n-2$, and so on. The right-hand side of the equation consists of the input at instants $k+m$, $k+m-1$, $k+m-2$, and so on.
- The condition that the above system is causal is $m \leq n$. For a general causal case, $m = n$, the above system can be expressed as

$$y[k+n] + a_{n-1}y[k+n-1] + \cdots + a_1 y[k+1] + a_0 y[k] =$$
$$b_n f[k+n] + b_{n-1} f[k+n-1] + \cdots + b_1 f[k+1] + b_0 f[k]$$

Discrete-Time System Equations

Delay operator form:

In case $m = n$, we can replace k by $k - n$ throughout the equation.

- Such replacement yields the delay operator form.

$$y[k] + a_{n-1}y[k-1] + \cdots + a_1 y[k-n+1] + a_0 y[k-n] =$$
$$b_n f[k] + b_{n-1} f[k-1] + \cdots + b_1 f[k-n+1] + b_0 f[k-n]$$

Iterative Solution of Difference Equations

From the delay operator form

$$y[k] + a_{n-1}y[k-1] + \cdots + a_1 y[k-n+1] + a_0 y[k-n] =$$
$$b_n f[k] + b_{n-1} f[k-1] + \cdots + b_1 f[k-n+1] + b_0 f[k-n]$$

It can be expressed as

$$y[k] = -a_{n-1}y[k-1] - a_{n-2}y[k-2] - \cdots - a_0 y[k-n]$$
$$+ b_n f[k] + b_{n-1} f[k-1] + \cdots + b_0 f[k-n]$$

There are the past n values of the output: $y[k-1]$, $y[k-2]$,. . ., $y[k-n]$, the past n values of the input: $f[k-1]$, $f[k-2]$, . . ., $f[k-n]$, and the present value of the input $f[k]$.

Initial Conditions and Iterative Solution of Difference Equations

If the input is causal, the $f[-1] = f[-2] = \ldots = f[-n] = 0$, and we need only n initial conditions $y[-1]$, $y[-2]$, …, $y[-n]$. This result allows us to compute iteratively or recursively the output $y[0]$, $y[1]$, $y[2]$, $y[3]$, …, and so on. For instant,

- to find $y[0]$ we set $k = 0$.
- the left-hand side is $y[0]$, and the right-hand side contains terms $y[-1], y[-2], \ldots, y[-n]$ and the input $f[0], f[-1], f[-2], \ldots, f[-n]$.
- Therefore, we must know the n initial conditions $y[-1], y[-2], \ldots, y[-n]$ to find $y[0]$, $y[1]$, $y[2]$, $\ldots$ and so on.

Iterative Solution of Difference Equations

Examples

Solve iteratively

$$y[k] - 0.5y[k-1] = f[k]$$

with initial condition $y[-1] = 16$ and causal input $f[k] = k^2$.

Solution:

Rewritten the equation in the delay operator form and move all past outputs to the left:

$$y[k] = 0.5y[k-1] + f[k]$$

We obtain

$$\begin{aligned}
y[0] &= 0.5y[-1] + f[0] = 0.5(16) + 0 = 8 \\
y[1] &= 0.5y[0] + f[1] = 0.5(8) + (1)^2 = 5 \\
y[2] &= 0.5y[1] + f[2] = 0.5(5) + (2)^2 = 6.5 \\
y[3] &= 0.5y[2] + f[3] = 0.5(6.5) + (3)^2 = 12.25 \\
y[4] &= 0.5y[3] + f[4] = 0.5(12.25) + (4)^2 = 22.125
\end{aligned}$$

Iterative Solution of Difference Equations

Examples cont.

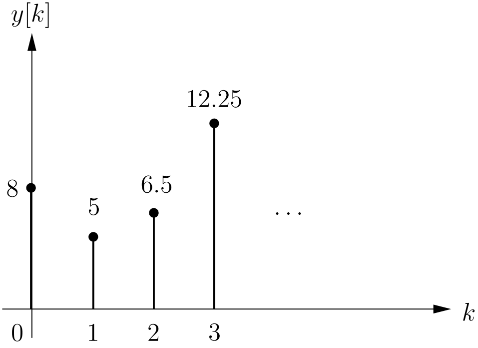


Figure: Iterative solution of a difference equation

Iterative Solution of Difference Equations

Examples cont.

Solve iteratively

$$y[k+2] - y[k+1] + 0.24y[k] = f[k+2] - 2f[k+1]$$

with initial conditions $y[-1] = 2$, $y[-2] = 1$ and a causal input $f[k] = k$.
Solution:
Rewritten the equation in the delay operator form and move all past outputs to the left:

$$y[k] = y[k-1] - 0.24y[k-2] + f[k] - 2f[k-1]$$

We obtain

$$y[0] = y[-1] - 0.24y[-2] + f[0] - 2f[-1] = 2 - 0.24(1) + 0 - 0 = 1.76$$
$$y[1] = y[0] - 0.24y[-1] + f[1] - 2f[0] = 1.76 - 0.24(2) + 1 - 0 = 2.28$$
$$y[2] = y[1] - 0.24y[0] + f[2] - 2f[1] = 2.28 - 0.24(1.76) + 2 - 2(1) = 1.8576$$
$$y[3] = y[2] - 0.24y[1] + f[3] - 2f[2] = 1.8576 - 0.24(2.28) + 3 - 2(2) = 0.3104$$
$$y[4] = y[3] - 0.24y[2] + f[4] - 2f[3] = 0.3104 - 0.24(1.8576) + 4 - 2(3) = -2.1354$$

Iterative Solution of Difference Equations

Examples cont.

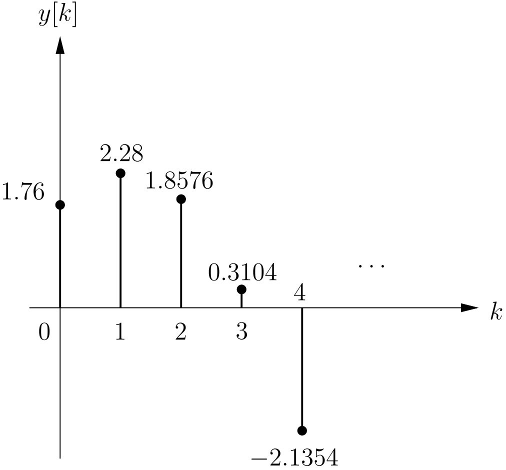


Figure: Iterative solution of a difference equation

E operator

In continuous-time system we used the operator D to denote the operation of differentiation. For discrete-time systems we use the operator E to denote the operation for advancing the sequence by one time unit. Thus

$$\begin{aligned} Ef[k] &= f[k+1] \\ E^2 f[k] &= f[k+2] \\ &\vdots \\ E^n f[k] &= f[k+n] \end{aligned}$$

For example

$$\begin{aligned} y[k+1] - ay[k] &= f[k+1] \\ Ey[k] - ay[k] &= Ef[k] \\ (E-a)y[k] &= Ef[k] \end{aligned}$$

E operator
cont.

For the second-order difference equation

$$y[k+2] + \frac{1}{4}y[k+1] + \frac{1}{16}y[k] = f[k+2]$$
$$\left(E^2 + \frac{1}{4}E + \frac{1}{16}\right) y[k] = E^2 f[k]$$

A general nth-order difference equation $(n = m)$can be expressed as

$$\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0\right) y[k] = \left(b_n E^n + b_{n-1}E^{n-1} + \cdots + b_1 E + b_0\right) f[k]$$
$$Q[E]y[k] = P[E]f[k]$$

where

$$Q[E] = E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0$$
$$P[E] = b_n E^n + b_{n-1}E^{n-1} + \cdots + b_1 E + b_0$$

Similar to the continuous-time case,

$$\boxed{\text{Total response} = \text{zero-input response} + \text{zero-state response}}$$

The zero-input response $y_0[k]$ is the solution of the system with $f[k] = 0$; that is,

$$Q[E]y_0[k] = 0$$

or

$$\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0\right) y_0[k] = 0$$

$$y_0[k+n] + a_{n-1}y_0[k+n-1] + \cdots + a_1 y_0[k+1] + a_0 y_0[k] = 0$$

The equation states that a linear combination of $y_0[k]$ and advanced $y_0[k]$ is zero not for some values of k but for all k. Such situation is possible if and only if $y_0[k]$ and advanced $y_0[k]$ have the same form. This is true only for an exponential function γ^k. Since

$$\gamma^{k+m} = \gamma^m \gamma^k$$

Therefore, if $y_0[k] = c\gamma^k$ we have

$$\begin{aligned} Ey_0[k] &= y_0[k+1] = c\gamma^{k+1} = c\gamma\gamma^k \\ E^2 y_0[k] &= y_0[k+2] = c\gamma^{k+2} = c\gamma^2\gamma^k \\ &\vdots \\ E^n y_0[k] &= y_0[k+n] = c\gamma^{k+n} = c\gamma^n\gamma^k \end{aligned}$$

Substitution of these results to the system equation yields

$$c\left(\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0\right)\gamma^k = 0$$

For a nontrivial solution of this equation

$$\left(\gamma^n + a_{n-1}\gamma^{n-1} + \cdots + a_1\gamma + a_0\right) = 0 \text{ or } Q[\gamma] = 0$$

$Q[\gamma]$ is an nth-order polynomial and can be expressed in the factorized form (assuming all distinct roots):

$$(\gamma - \gamma_1)(\gamma - \gamma_2)\cdots(\gamma - \gamma_n) = 0$$

Clearly, γ has n solutions $\gamma_1, \gamma_2, \cdots, \gamma_n$ and, the system has n solutions $c_1\gamma_1^k, c_2\gamma_2^k, \ldots, c_n\gamma_n^k$.

The zero-input response is

$$y_0[k] = c_1\gamma_1^k + c_2\gamma_2^k + \cdots + c_n\gamma_n^k$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the roots of the polynomial.

- $Q[\gamma]$ is called the **characteristic polynomial** of the system.
- $Q[\gamma] = 0$ is the **characteristic equation** of the system.
- $\gamma_1, \gamma_2, \ldots, \gamma_n$ are called **characteristic roots** or **characteristic values** (also **eignevalues**) of the system.
- The exponentials $\gamma_i^k (i = 1, 2, \ldots, n)$ are **characteristic modes** or **natural modes** of the system.

Repeated Roots:

If two or more roots are repeated, the form of the characteristic modes is modified. Similar to the continuous-time case, if a root γ repeats r times, the characteristic modes corresponding to this root are γ^k, $k\gamma^k$, $k^2\gamma^k$, ..., $k^{r-1}\gamma^k$.

If the characteristic equation of a system is

$$Q[\gamma] = (\gamma - \gamma_1)^r(\gamma - \gamma_{r+1})(\gamma - \gamma_{r+2})\cdots(\gamma - \gamma_n)$$

the zero-input response of the system is

$$y_0[k] = (c_1 + c_2 k + c_3 k^2 + \cdots + c_r k^{r-1})\gamma_1^k + c_{r+1}\gamma_{r+1}^k + c_{r+2}\gamma_{r+2}^k + \cdots + c_n\gamma_n^k$$

Complex Roots:

As in the case of continuous-time systems, the complex roots of a discrete-time system must occur in pairs of conjugates so that the system equation coefficients are real. Like the case of continuous-time systems, we can eliminate dealing with complex numbers by using the real form of the solution.

- First express the complex conjugate roots γ and γ^* in polar form.

$$\gamma = |\gamma|e^{j\beta} \text{ and } \gamma^* = |\gamma|e^{-j\beta}$$

- the zero-input response is given by

$$\begin{aligned} y_0[k] &= C_1\gamma^k + C_2(\gamma^*)^k \\ &= C_1|\gamma|^k e^{j\beta k} + C_2|\gamma|^k e^{-j\beta k} \end{aligned}$$

For a real system, C_1 and C_2 must be conjugates so that $y_0[k]$ is a real function of k. Let

$$C_1 = \frac{C}{2}e^{j\theta} \text{ and } C_2 = \frac{C}{2}e^{-j\theta}$$

$$\begin{aligned} y_0[k] &= \frac{C}{2}|\gamma|^k \left[e^{j(\beta k+\theta)} + e^{-j(\beta k+\theta)}\right] \\ &= C|\gamma|^k \cos(\beta k + \theta) \end{aligned}$$

where C and θ are arbitrary constants determined from the auxiliary conditions.

For an LTID system described by the difference equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

Find the zero-input response $y_0[k]$ of the system if the initial conditions are $y[-1] = 0$ and $y[-2] = \frac{25}{4}$.

The system equation in E operator form is

$$(E^2 - 0.6E - 0.16)y[k] = 5E^2 f[k]$$

The characteristic equation is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma + 0.8) = 0$$

The zero-input response is

$$y_0[k] = C_1(-0.2)^k + C_2(0.8)^k$$

Substitute $y_0[-1] = 0$ and $y_0[-2] = \frac{25}{4}$ we obtain

$$-5C_1 + \frac{5}{4}C_2 = 0$$
$$25C_1 + \frac{25}{16}C_2 = \frac{25}{4}$$

and $C_1 = \frac{1}{5}$ and $C_2 = \frac{4}{5}$. Therefore

$$y_0[k] = \frac{1}{5}(-0.2)^k + \frac{4}{5}(0.8)^k, \; k \geq 0.$$

A system specified by the equation

$$(E^2 + 6E + 9)y[k] = (2E^2 + 6E)f[k]$$

determine $y_0[k]$, the zero-input response, if the initial condition are $y_0[-1] = -\frac{1}{3}$ and $y_0[-2] = -\frac{2}{9}$.
The characteristic equation is

$$\gamma^2 + 6\gamma + 9 = (\gamma + 3)^2 = 0$$

and we have a repeated characteristic root at $\gamma = -3$. Hence, the zero-input response is

$$y_0[k] = (C_1 + C_2 k)(-3)^k.$$

From the initial conditions we have

$$C_1 - C_2 = 1$$
$$C_1 - 2C_2 = -2$$

and $C_1 = 4$, $C_2 = 3$. Finally, we have $y_0[k] = (4 + 3k)(-3)^k$, $k \geq 0$.

Find the zero-input response of an LTID system described by the equation

$$(E^2 - 1.56E + 0.81)y[k] = (E+3)f[k]$$

when the initial conditions are $y_0[-1] = 2$ and $y_0[-2] = 1$.
The characteristic equation is

$$(\gamma^2 - 1.56\gamma + 0.81) = (\gamma - 0.78 - j0.45)(\gamma - 0.78 + j0.45) = 0.$$

The characteristic roots are $0.78 \pm j0.45$; that is , $0.9e^{\pm j\frac{\pi}{6}}$. Thus, $|\gamma| = 0.9$ and $\beta = \frac{\pi}{6}$, and the zero-input response is given by

$$y_0[k] = C(0.9)^k \cos(\frac{\pi}{6}k + \theta).$$

Substituting the initial conditions $y_0[-1] = 2$ and $y_0[-2] = 1$, we obtain

$$\frac{C}{0.9}\cos\left(-\frac{\pi}{6} + \theta\right) = \frac{C}{0.9}\left[\cos(-\frac{\pi}{6})\cos\theta - \sin(-\frac{\pi}{6})\sin\theta\right] = \frac{C}{0.9}\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right] = 2$$

and

$$\frac{C}{(0.9)^2}\cos\left(-\frac{\pi}{3}+\theta\right)=\frac{C}{0.81}\left[\cos(-\frac{\pi}{3})\cos\theta-\sin(-\frac{\pi}{3})\sin\theta\right]=\frac{C}{0.81}\left[\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right]=1$$

or

$$\frac{\sqrt{3}}{1.8}C\cos\theta+\frac{1}{1.8}C\sin\theta=2$$
$$\frac{1}{1.62}C\cos\theta+\frac{\sqrt{3}}{1.62}C\sin\theta=1.$$

We have $C\cos\theta=2.308$ and $C\sin\theta=-0.397$. Then

$$\theta=\tan^{-1}\frac{-0.397}{2.308}=-0.17 \text{ rad}$$

Substituting $\theta=-0.17$ radian in $C\cos\theta=2.308$ yields $C=2.34$ and

$$y_0[k]=2.34(0.9)^k\cos\left(\frac{\pi}{6}k-0.17\right),\ k\geq 0$$

The Unit Impulse Response $h[k]$

Consider an nth-order system specified by the equation

$$\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1 E + a_0\right) y[k] = \left(b_n E^n + b_{n-1}E^{n-1} + \cdots + b_1 E + b_0\right) f[k]$$

or

$$Q[E]y[k] = P[E]f[k]$$

The unit impulse response $h[k]$ is the solution of this equation for the input $\delta[k]$ with all the initial conditions zero; that is

$$Q[E]h[k] = P[E]\delta[k]$$

subject to initial conditions

$$h[-1] = h[-2] = \cdots = h[-n] = 0$$

The Unit Impulse Response $h[k]$

$h[k]$ is the system response to the input $\delta[k]$, which is zero for $k > 0$. We know that when the input is zero, only the characteristic modes can be sustained by the system. Therefore, $h[k]$ must be made up of characteristic modes for $k > 0$. At $k = 0$, it may have some nonzero value, and $h[k]$ can be expressed as

$$h[k] = \frac{b_0}{a_0}\delta[k] + y_n[k]u[k].$$

The n unknown coefficients in $y_n[k]$ can be determined from a knowledge of n values of $h[k]$. It is a straightforward task to determine values of $h[k]$ iteratively.

For a discrete-time system specified above, we have

$$h[k] = A_0\delta[k] + y_n[k]u[k]$$

Then

$$Q[E](A_0\delta[k] + y_n[k]u[k]) = P[E]\delta[k]$$

because $y_n[k]u[k]$ is a sum of characteristic modes

$$Q[E](y_n[k]u[k]) = 0,\ k \geq 0$$

The above equation reduces to

$$A_0 Q[E]\delta[k] = P[E]\delta[k],\ k \geq 0$$

or

$$A_0\left(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0\right)\delta[k] = \left(b_nE^n + b_{n-1}E^{n-1} + \cdots + b_1E + b_0\right)\delta[k]$$

$$A_0(\delta[k+n] + a_{n-1}\delta[k+n-1] + \cdots + a_1\delta[k+1] + a_0\delta[k]) = b_n\delta[k+n] + b_{n-1}\delta[k+n-1] + \cdots + b_1\delta[k+1] + b_0\delta[k]$$

If we set $k = 0$ in the equation and recognize that $\delta[0] = 1$ and $\delta[m] = 0$ when $m \neq 0$, all but the last terms vanish on both sides, yielding

$$A_0a_0 = b_0 \text{ and } A_0 = \frac{b_0}{a_0}$$

Note: for the special case $a_0 = 0$ see the reference.

The Unit Impulse Response $h[k]$

Example

Determine the unit impulse response $h[k]$ for a system specified by the equation

$$y[k] - 0.6y[k-1] - 0.16y[k-2] = 5f[k]$$

This equation can be expressed in the advance operator form as

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

or

$$\left(E^2 - 0.6E - 0.16\right) y[k] = 5E^2 f[k]$$

The characteristic equation is

$$\gamma^2 - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8) = 0.$$

Therefore

$$y_n[k] = C_1(-0.2)^k + C_2(0.8)^k$$

The Unit Impulse Response $h[k]$

Example cont.

From the system we have $a_0 = -0.16$ and $b_0 = 0$. Therefore

$$h[k] = \frac{0}{-0.16}\delta[k] + \left[C_1(-0.2)^k + C_2(0.8)^k\right]u[k] = \left[C_1(-0.2)^k + C_2(0.8)^k\right]u[k]$$

To determine C_1 and C_2, we need to find two values of $h[k]$ iteratively. To do this, we must let the input $f[k] = \delta[k]$ and the output $y[k] = h[k]$ in the system equation. The resulting equation is

$$h[k] - 0.6h[k-1] - 0.16h[k-2] = 5\delta[k]$$

subject to zero initial state; that is , $h[-1] = h[-2] = 0$.
Setting $k = 0$ in this equation yields

$$h[0] - 0.6(0) - 0.16(0) = 5(1) \Longrightarrow h[0] = 5$$

Next, setting $k = 1$ and using $h[0] = 5$, we obtain

The Unit Impulse Response $h[k]$

Example cont.

Then we have

$$h[0] = C_1(-0.2)^0 + C_2(0.8)^0 = C_1 + C_2 = 5$$
$$h[1] = C_1(-0.2)^1 + C_2(0.8)^1 = -0.2C_1 + 0.8C_2 = 3$$

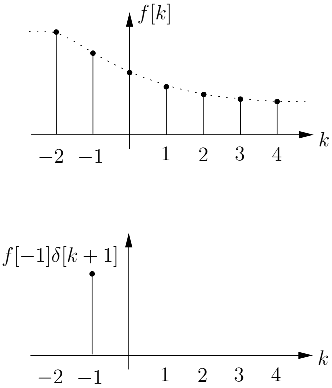
and $C_1 = 1$, $C_2 = 4$. Therefore

$$h[k] = \left[(-0.2)^k + 4(0.8)^k\right] u[k]$$

System Response to External Input

The Zero-State Response

The zero-state response $y[k]$ is the system response to an input $f[k]$ when the system is in zero state. Like in the continuous-time case an arbitrary input $f[k]$ can be expressed as a sum of impulse components.



The previous page shows how a signal $f[k]$ can be expressed as a sum of impulse components. The component of $f[k]$ at $k = m$ is $f[m]\delta[k-m]$, and $f[k]$ is the sum of all these components summed from $m = -\infty$ to ∞. Therefore

$$
\begin{aligned}
f[k] &= f[0]\delta[k] + f[1]\delta[k-1] + f[2]\delta[k-2] + \cdots \\
&\qquad + f[-1]\delta[k+1] + f[-2]\delta[k+2] + \cdots \\
&= \sum_{m=-\infty}^{\infty} f[m]\delta[k-m]
\end{aligned}
$$

If we knew the system response to impulse $\delta[k]$, the system response to any arbitrary input could be obtained by summing the system response to various impulse components.

System Response to External Input

The Zero-State Response cont.

If

$$\delta[k] \implies h[k]$$

then

$$\begin{aligned} \delta[k-m] &\implies h[k-m] \\ f[m]\delta[k-m] &\implies f[m]h[k-m] \\ \underbrace{\sum_{m=-\infty}^{\infty} f[m]\delta[k-m]}_{f[k]} &\implies \underbrace{\sum_{m=-\infty}^{\infty} f[m]h[k-m]}_{y[k]} \end{aligned}$$

System Response to External Input

The Zero-State Response cont.

We have the response $y[k]$ to input $f[k]$ as

$$y[k] = \sum_{m=-\infty}^{\infty} f[m]h[k-m].$$

This summation on the right-had side is known as the **convolution sum** of $f[k]$ and $h[k]$, and is represented symbolically by $f[k] * h[k]$

$$f[k] * h[k] = \sum_{m=-\infty}^{\infty} f[m]h[k-m]$$

The Commutative Property

$$f_1[k] * f_2[k] = f_2[k] * f_1[k]$$

This can be proved as follow:

$$\begin{aligned} f_1[k] * f_2[k] &= \sum_{m=-\infty}^{\infty} f_1[m] f_2[k-m] \\ &= -\sum_{w=\infty}^{-\infty} f_1[w-k] f_2[w], \qquad w = k - m \\ &= \sum_{w=-\infty}^{\infty} f_2[w] f_1[w-k] \\ &= f_2[k] * f_1[k] \end{aligned}$$

The Distributive Property

$$f_1[k] * (f_2[k] + f_3[k]) = f_1[k] * f_2[k] + f_1[k] * f_3[k]$$

The proof is as follow:

$$\begin{aligned} f_1[k] * (f_2[k] + f_3[k]) &= \sum_{m=-\infty}^{\infty} f_1[m] \left(f_2[k-m] + f_3[k-m] \right) \\ &= \sum_{m=-\infty}^{\infty} f_1[m] f_2[k-m] + \sum_{m=-\infty}^{\infty} f_1[m] f_3[k-m] \\ &= f_1[k] * f_2[k] + f_1[k] * f_3[k] \end{aligned}$$

The Associative Property

$$f_1[k] * (f_2[k] * f_3[k]) = (f_1[k] * f_2[k]) * f_3[k]$$

The proof is as follow:

$$\begin{aligned} f_1[k] * (f_2[k] * f_3[k]) &= \sum_{m_1=-\infty}^{\infty} f_1[m_1]\,(f_2[k-m_1] * f_3[k-m_1]) \\ &= \sum_{m_1=-\infty}^{\infty} f_1[m_1] \sum_{m_2=-\infty}^{\infty} f_2[m_2] f_3[k-m_1-m_2] \\ &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_1[\lambda-m_2] f_2[m_2] f_3[k-\lambda] \end{aligned}$$

,where $\lambda = m_1 + m_2$.

Then we have

$$f_1[k] * (f_2[k] * f_3[k]) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} f_1[\lambda - m_2] f_2[m_2] f_3[k - \lambda]$$
$$= (f_1[k] * f_2[k]) * f_3[k]$$

The Convolution with an Impulse

$$f[k] * \delta[k] = \sum_{m=-\infty}^{\infty} f[m] \delta[k - m]$$

Since $\delta[k-m] = 1$, if $k - m = 0$ or $m = k$, then

$$f[k] * \delta[k] = f[k].$$

The shifting Property

If

$$f_1[k] * f_2[k] = c[k]$$

then

$$\begin{aligned} f_1[k] * f_2[k-n] &= f_1[k] * f_2[k] * \delta[k-n] \\ &= c[k] * \delta[k-n] = c[k-n] \\ f_1[k-n] * f_2[k] &= f_1[k] * \delta[k-n] * f_2[k] \\ &= f_1[k] * f_2[k] * \delta[k-n] \\ &= c[k] * \delta[k-n] = c[k-n] \\ f_1[k-n] * f_2[k-l] &= f_1[k] * \delta[k-n] * f_2[k] * \delta[k-l] \\ &= c[k] * \delta[k-n] * \delta[k-l] = c[k-n-l] \end{aligned}$$

The Width Property

If $f_1[k]$ and $f_2[k]$ have lengths of m and n elements respectively, then the length of $c[k]$ is $m+n-1$ elements.

Causality and Zero-State Response

- We assumed the system to be linear and time-invariant.
- In practice, almost all of the input signals are causal, and a majority of the system are also causal.
- If the input $f[k]$ is causal, then $f[m] = 0$ for $m < 0$.
- Similarly, if the system is causal, then $h[x] = 0$ for negative x, so that $h[k-m] = 0$ when $m > k$.
- Therefore, if $f[k]$ and $h[k]$ are both causal, the product $f[m]h[k-m] = 0$ for $m < 0$ and for $m > k$, and it is nonzero only for the range $0 \leq m \leq k$. Therefore, the convolution sum is reduced to

$$y[k] = \sum_{m=0}^{k} f[m]h[k-m]$$

Convolution Sum

Analytical Method Example

Determine $c[k] = f[k] * g[k]$ for

$$f[k] = (0.8)^k u[k] \;\text{ and }\; g[k] = (0.3)^k u[k]$$

we have

$$c[k] = \sum_{m=0}^{k} f[m]g[k-m]$$

since both signals are causal.

$$c[k] = \begin{cases} \displaystyle\sum_{m=0}^{k} (0.8)^m (0.3)^{k-m} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$c[k] = (0.3)^k \sum_{m=0}^{k} \left(\frac{0.8}{0.3}\right)^m u[k] = (0.3)^k \frac{(0.8)^{k+1} - (0.3)^{k+1}}{(0.3)^k (0.8 - 0.3)} u[k]$$

$$= 2\left[(0.8)^{k+1} - (0.3)^{k+1}\right] u[k]$$

Zero-State Response

Analytical Method Example

Find the zero-state response $y[k]$ of an LTID system described by the equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

if the input $f[k] = 4^{-k}u[k]$ and $h[k] = \left[(-0.2)^k + 4(0.8)^k\right]u[k]$.
We have

$$\begin{aligned}
y_s[k] &= f[k] * h[k] \\
&= (4)^{-k}u[k] * \left[(-0.2)^k u[k] + 4(0.8)^k u[k]\right] \\
&= (4)^{-k}u[k] * (-0.2)^k u[k] + (4)^{-k}u[k] * 4(0.8)^k u[k] \\
&= (0.25)^k u[k] * (-0.2)^k u[k] + 4(0.25)^k u[k] * (0.8)^k u[k]
\end{aligned}$$

Using Pair 4 from the convolution sum table:

$$y[k] = \left[\frac{(0.25)^{k+1} - (-0.2)^{k+1}}{0.25 - (-0.2)} + 4\frac{(0.25)^{k+1} - (0.8)^{k+1}}{0.25 - 0.8}\right]u[k]$$

$$y[k] = \left(2.22\left[(0.25)^{k+1} - (-0.2)^{k+1}\right] - 7.27\left[(0.25)^{k+1} - (0.8)^{k+1}\right]\right)u[k]$$
$$= \left[-5.05(0.25)^{k+1} - 2.22(-0.2)^{k+1} + 7.27(0.8)^{k+1}\right]u[k]$$

Recognizing that

$$\gamma^{k+1} = \gamma(\gamma)^k$$

We can express $y[k]$ as

$$y[k] = \left[-1.26(0.25)^k + 0.444(-0.2)^k + 5.81(0.8)^k\right]u[k]$$
$$= \left[-1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k\right]u[k]$$

Graphical Procedure for the Convolution Sum

The convolution sum of causal signals $f[k]$ and $g[k]$ is given by

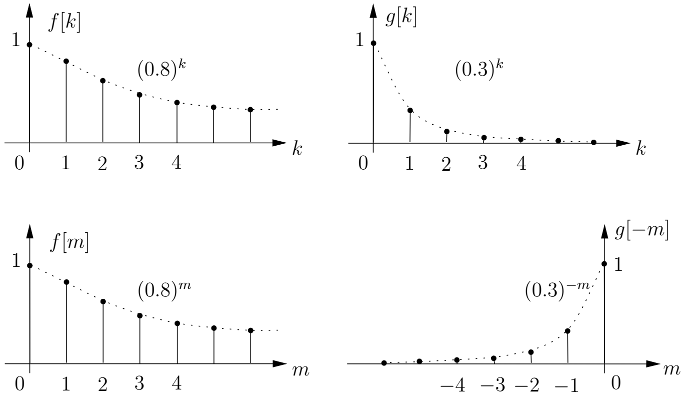
$$c[k] = \sum_{m=0}^{k} f[m]g[k-m]$$

- Invert $g[m]$ about the vertical axis $(m = 0)$ to obtain $g[-m]$.
- Time shift $g[-m]$ by k units to obtain $g[k-m]$. For $k > 0$, the shift is to the right (delay); for $k < 0$, the shift is to the left (advance).
- Next we multiply $f[m]$ and $g[k-m]$ and add all the products to obtain $c[k]$. The procedure is repeated to each value of k over the range $-\infty$ to ∞.

Graphical Procedure for the Convolution Sum

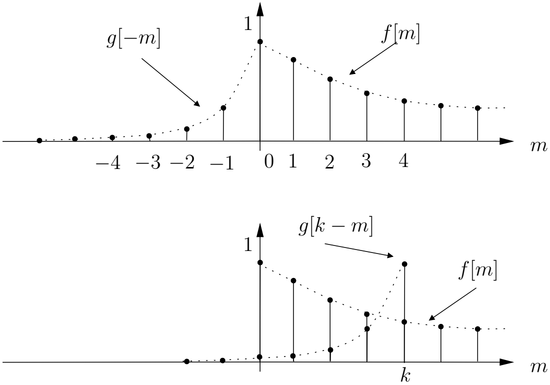
Example

Find $c[k] = f[k] * g[k]$, where $f[k]$ and $g[k]$ are depicted in the Figures.



Graphical Procedure for the Convolution Sum

Example



The two functions $f[m]$ and $g[k-m]$ overlap over the interval $0 \leq m \leq k$.

Therefore

$$
\begin{aligned}
c[k] &= \sum_{m=0}^{k} f[m]g[k-m] \\
&= \sum_{m=0}^{k} (0.8)^m (0.3)^{k-m} \\
&= (0.3)^k \sum_{m=0}^{k} \left(\frac{0.8}{0.3}\right)^m \\
&= 2\left[(0.8)^{k+1} - (0.3)^{k+1}\right], \qquad k \geq 0
\end{aligned}
$$

For $k < 0$, there is no overlap between $f[m]$ and $g[k-m]$, so that $c[k] = 0 \qquad k < 0$ and

$$c[k] = 2\left[(0.8)^{k+1} - (0.3)^{k+1}\right] u[k].$$

Graphical Procedure for the Convolution Sum

Sliding Tape Method

Using the sliding tape method, convolve the two sequences $f[k]$ and $g[k]$.

- write the sequences $f[k]$ and $g[k]$ in the slots of two tapes
- leave the f tape stationary (to correspond to $f[m]$). The $g[-m]$ tape is obtained by time inverting the $g[m]$
- shift the inverted tape by k slots, multiply values on two tapes in adjacent slots, and add all the products to find $c[k]$.

For the case of $k = 0$,

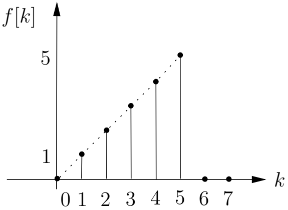
$$c[0] = 0 \times 1 = 0$$

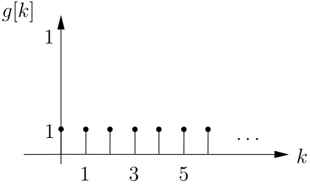
For $k = 1$

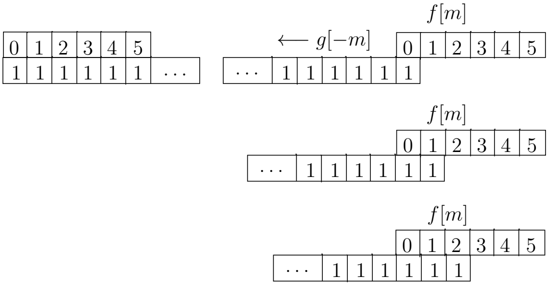
$$c[1] = (0 \times 1) + (1 \times 1) = 1$$

Similarly,

$$\begin{aligned}
c[2] &= (0 \times 1) + (1 \times 1) + (2 \times 1) = 3 \\
c[3] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) = 6 \\
c[4] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) = 10 \\
c[5] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) + (5 \times 1) = 15 \\
c[6] &= (0 \times 1) + (1 \times 1) + (2 \times 1) + (3 \times 1) + (4 \times 1) + (5 \times 1) = 15
\end{aligned}$$







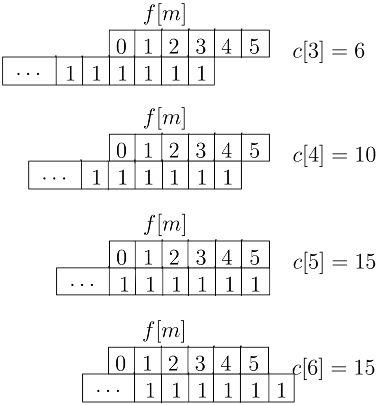
$c[0] = 0$

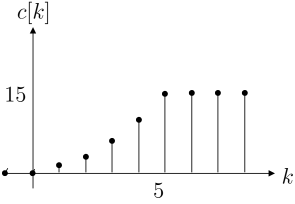
$c[1] = 1$

$c[2] = 3$

Graphical Procedure for the Convolution Sum

Sliding Tape Method cont.





Total Response

The total response of an LTID system can be expressed as a sum of the zero-input and zero-state components:

$$\text{Total response } y[k] = \underbrace{\sum_{j=1}^{n} c_j \gamma_j^k}_{\text{Zero-input component}} + \underbrace{f[k] * h[k]}_{\text{Zero-state component}}$$

From the previous example, the system described by the equation

$$y[k+2] - 0.6y[k+1] - 0.16y[k] = 5f[k+2]$$

with initial conditions $y[-1] = 0, y[-2] = \frac{25}{4}$ and input $f[k] = (4)^{-k}u[k]$. We have

$$y[k] = \underbrace{0.2(-0.2)^k + 0.8(0.8)^k}_{\text{Zero-input component}} \underbrace{-1.26(4)^{-k} + 0.444(-0.2)^k + 5.81(0.8)^k}_{\text{Zero-state component}}$$

Classical solution of Linear Difference Equations

If $y_n[k]$ and $y_\phi[k]$ denote the natural and the forced response respectively, the the total response is given by

$$\text{Total response } y[k] = \underbrace{y_n[k]}_{\text{modes}} + \underbrace{y_\phi[k]}_{\text{nonmodes}}$$

Because $y_n[k] + y_\phi[k]$ is a solution of the system, we have

$$Q[E](y_n[k] + y_\phi[k]) = P[E]f[k]$$

$y_n[k]$ is made up of characteristic modes,

$$Q[E]y_n[k] = 0$$

Substitution of this equation yields

$$Q[E]y_\phi[k] = P[E]f[k]$$

Classical solution of Linear Difference Equations

Forced Response

By definition, the forced response contains only nonmode terms and and the list of the inputs and the corresponding forms of the forced function is show below:

Input $f[k]$	Forced Response $y_\phi[k]$
1. $r^k \; , r \neq \gamma_i \; (i = 1, 2, \cdots, n)$	cr^k
2. $r^k \; , r = \gamma_i$	ckr^k
3. $\cos(\beta k + \theta)$	$c\cos(\beta k + \phi)$
4. $\left(\sum_{i=0}^{m} \alpha_i k^i\right) r^k$	$\left(\sum_{i=0}^{m} c_i k^i\right) r^k$

Note: By definition $y_\phi[k]$ cannot have any characteristic mode terms.

Determine the total response $y[k]$ of a system

$$(E^2 - 5E + 6)y[k] = (E - 5)f[k]$$

if the input $f[k] = (3k + 5)u[k]$ and the auxiliary conditions are $y[0] = 4, y[1] = 13$.
The characteristic equation is

$$\gamma^2 - 5\gamma + 6 = (\gamma - 2)(\gamma - 3) = 0$$

Therefore, the natural response is

$$y_n[k] = B_1(2)^k + B_2(3)^k$$

To find the form of forced response $y_\phi[k]$, we use above Table, Pair 4 with $r = 1, m = 1$.
This yields

$$y_\phi[k] = c_1 k + c_0$$

Classical solution of Linear Difference Equations

Forced Response example cont.

Therefore

$$y_\phi[k+1] = c_1(k+1) + c_0 = c_1 k + c_1 + c_0$$
$$y_\phi[k+2] = c_1(k+2) + c_0 = c_1 k + 2c_1 + c_0$$

Also

$$f[k] = 3k + 5$$

and

$$f[k+1] = 3(k+1) + 5 = 3k + 8$$

Substitution of the above results yiels

$$c_1 k + 2c_1 + c_0 - 5(c_1 k + c_1 + c_0) + 6(c_1 k + c_0) = 3k + 8 - 5(3k+5)$$
$$2c_1 k - 3c_1 + 2c_0 = -12k - 17$$

Classical solution of Linear Difference Equations

Forced Response example cont.

Comparison of similar terms on the two sides yields

$$\begin{aligned} 2c_1 &= -12 \\ -3c_1 + 2c_0 &= -17 \end{aligned}$$

and $c_1 = -6,\ c_0 = -\frac{35}{2}$. Therefore

$$y_\phi[k] = -6k - \frac{35}{2}$$

The total response is

$$\begin{aligned} y[k] &= y_n[k] + y_\phi[k] \\ &= B_1(2)^k + B_2(3)^k - 6k - \frac{35}{2},\ k \geq 0 \end{aligned}$$

Classical solution of Linear Difference Equations

Forced Response example cont.

To determine arbitrary constants B_1 and B_2 we set $k = 0$ and 1 and substitute the initial conditions $y[0] = 4,\ y[1] = 13$ to obtain

$$B_1 + B_2 - \frac{35}{2} = 4$$
$$2B_1 + 3B_2 - \frac{47}{2} = 13$$

and $B_1 = 28,\ B_2 = -\frac{13}{2}$.
Therefore

$$y_n[k] = 28(2)^k - \frac{13}{2}(3)^k$$

and

$$y[k] = \underbrace{28(2)^k - \frac{13}{2}(3)^k}_{y_n[k]} \underbrace{-6k - \frac{35}{2}}_{y_\phi[k]}$$

Reference

1. Lathi, B. P., *Signal Processing & Linear Systems*,
Berkeley-Cambridge Press, 1998.