Lecture 6: Discrete-Time Signals and Systems

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Outline

- Introduction
- Some Useful Discrete-Time Signal Models
- Size of a Discrete-Time Signal
- Useful Signal Operations
- Examples

Introduction

- Signals defined only at discrete instants of time are **discrete-time signals**.
- We consider uniformly spaced discrete instants such as $\ldots, -2T, -T, 0, T, 2T, 3T, \ldots, kT, \ldots$.
- Discrete-time signals can be specified as $f(kT), y(kT)$, where k are integer.
- Frequently used notation are $f[k], y[k]$, etc., where they are understood that $f[k] = f(kT), y[k] = y(kT)$ and that k are integers.
- Typical discrete-time signals are just sequences of numbers.
- a discrete-time system may seen as processing a sequence of numbers $f[k]$ and yielding as output another sequence of numbers $y[k]$.

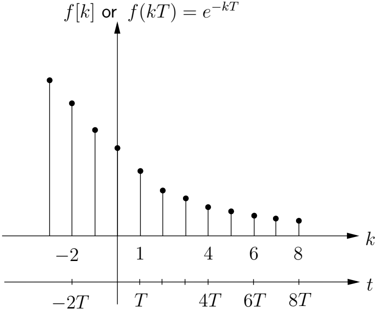
Discrete-time signal

Example

a continuous-time exponential $f(t) = e^{-t}$, when sampled every $T = 0.1$ second, results in a discrete-time signal $f(kT)$ given by

$$f(kT) = e^{-kT} = e^{-0.1k}$$

This is a function of k and may be expressed as $f[k]$.



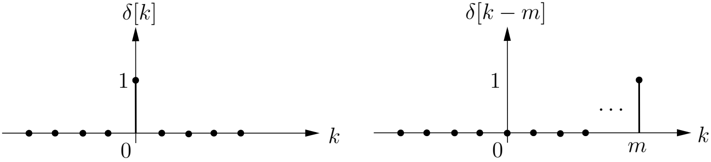
Discrete-time signal

Example

- Discrete-time signals arise naturally in situations which are inherently discrete-time
- such as population studies, amortization problems, national income models, and radar tracking.
- They may also arise as a result of sampling continuous-time signals in sampled data systems, digital filtering, etc.

The discrete-time counterpart of the continuous-time impulse function $\delta(t)$ is $\delta[k]$, defined by

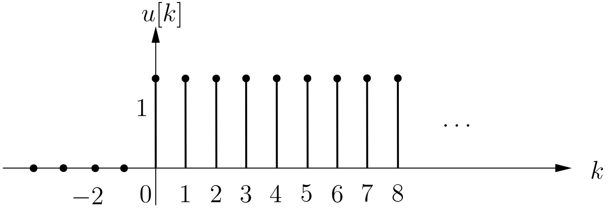
$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$



Unlike its continuous-time counterpart $\delta(t)$, this is a very simple function without any mystery.

The discrete-time counterpart of the unit step function $u(t)$ is $u[k]$, defined by

$$u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



If we want a signal to start at $k = 0$, we need only multiply the signal with $u[k]$.

Some Useful Discrete-time signal Models

Discrete-Time Exponential γ^k

- a continuous-time exponential $e^{\lambda t}$ can be expressed in an alternate form as

$$e^{\lambda t} = \gamma^t \qquad (\gamma = e^{\lambda} \text{ or } \lambda = \ln \gamma)$$

  For example,

  - $e^{-0.3t} = (e^{-0.3})^t = (0.7408)^t$
  - $4^t = e^{1.386t}$ because $\ln 4 = 1.386$ that is $e^{1.386} = 4$

- In the study of continuous-time signals and systems we prefer the form $e^{\lambda t}$ rather that γ^t.

Some Useful Discrete-time signal Models

Discrete-Time Exponential γ^k cont.

- The discrete-time exponential can also be expressed in two form as

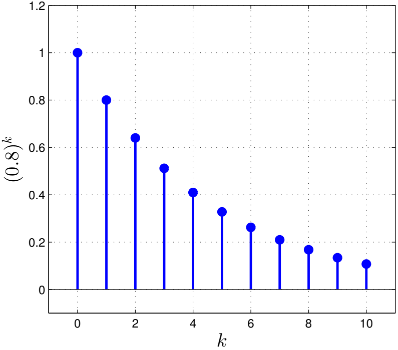
$$e^{\lambda k} = \gamma^k \qquad (\gamma = e^{\lambda} \text{ or } \lambda = \ln \gamma)$$

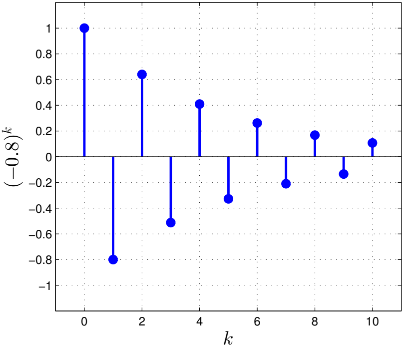
  For example

  - $e^{3k} = (e^3)^k = (20.086)^k$
  - $5^k = (e^{1.609})^k$ because $5 = e^{1.609}$

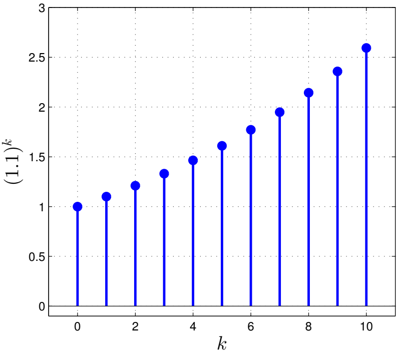
- In the study of discrete-time signals and systems, the form γ^k proves more convenient than the form $e^{\lambda k}$

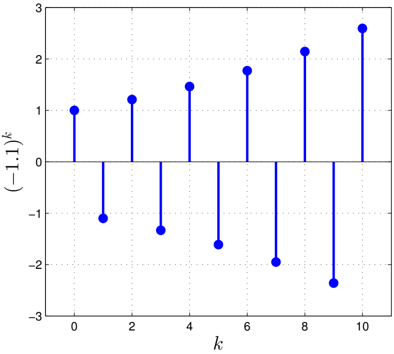
- If $|\gamma| = 1$, then $\cdots = \gamma^{-1} = \gamma^0 = \gamma^1 = \cdots = 1$
- If $|\gamma| < 1$, then the signal decays exponentially with k.



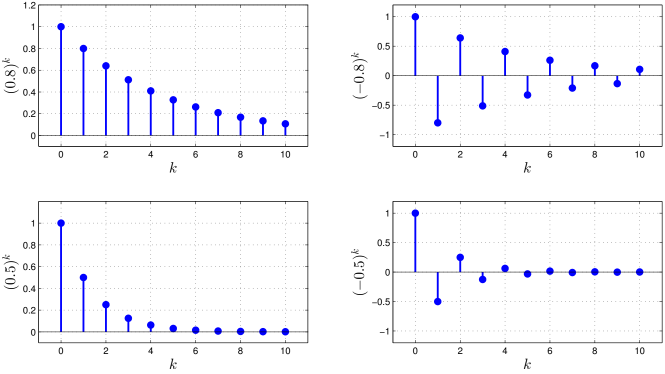


- If $|\gamma| > 1$, then the signal grows exponentially with k.

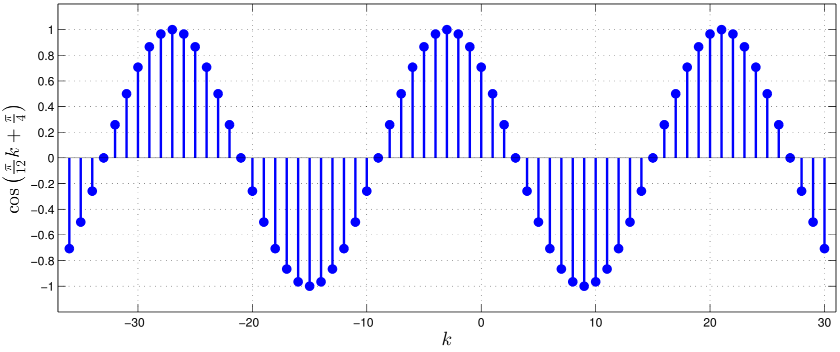




- the exponential $(0.5)^k$ decays faster than $(0.8)^k$
- $\gamma^{-k} = \left(\dfrac{1}{\gamma}\right)^k$ for example the exponential $(0.5)^k$ can be expressed as 2^{-k}.



- A general discrete-time sinusoid can be expressed as $C\cos(\Omega k + \theta)$, where C is the **amplitude**, Ω is the **frequency** (in radians per sample), and θ is the **phase** (in radians)



Some Useful Discrete-time signal Models

Discrete-Time Sinusoid $\cos(\Omega k + \theta)$ cont.

- Because $\cos(-x) = \cos(x)$

$$\cos(-\Omega k + \theta) = \cos(\Omega k - \theta)$$

  It shows that both $\cos(\Omega k + \theta)$ and $\cos(-\Omega k + \theta)$ have the same frequency (Ω). Therefor, the frequency of $\cos(\Omega k + \theta)$ is $|\Omega|$.

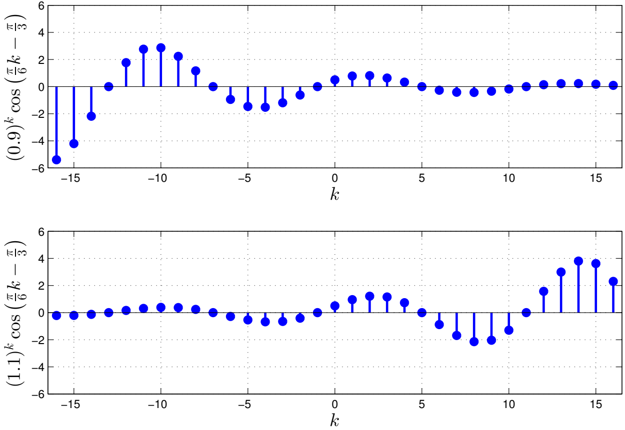
- Sampled continuous-time sinusoid yields a discrete-time sinusoid

$$f[k] = \cos \omega k T = \cos \Omega k \qquad \text{where } \Omega = \omega T$$

Some Useful Discrete-time signal Models

Exponentially varying discrete-time sinusoid $\gamma^k \cos(\Omega k + \theta)$

- $\gamma^k \cos(\Omega k + \theta)$ is a sinusoid $\cos(\Omega k + \theta)$ with an exponentially varying amplitude γ^k.



Size of a Discrete-Time Signal

Energy signal

- The size of a discrete-time signal $f[k]$ can be measured by its energy E_f defined by

$$E_f = \sum_{k=-\infty}^{\infty} |f[k]|^2$$

- the measure is meaningful if the energy of a signal is finite. A necessary condition for the energy to be finite is that the signal amplitude must approach 0 as $|k| \to \infty$. Otherwise the sum will not converge.
- If E_f is finite, the signal is called an **energy signal**.

Size of a Discrete-Time Signal

Power signal

- For the cases, the amplitude of $f[k]$ does not approach to 0 as $|k| \to \infty$, then the signal energy is infinite, and a measure of the signal will be the time average of the energy (if it exists).
- the signal power P_f is defined by

$$P_f = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |f[k]|^2$$

- For periodic signals, the time averaging need be performed only over one period in view of the periodic repetition of the signal.
- If P_f is finite and nonzero, called a **power signal**.
- A discrete-time signal can either be an energy signal or a power signal. Some signals are neither energy nor power signals.

Size of a Discrete-Time Signal
Example

Show that the signal $a^k u[k]$ is an energy signal of energy $\frac{1}{1-|a|^2}$ if $|a| < 1$. It is a power signal of power $P_f = 0.5$ if $|a| = 1$. It is neither an energy signal nor a power signal if $|a| > 1$.
Solution:

$$E_f = \sum_{k=-\infty}^{\infty} |f[k]|^2 = \sum_{k=0}^{\infty} |a^k|^2$$

If $|a| < 1$ then $|a^k|$ approaches to 0 and

$$\begin{aligned} S &= \sum_{k=0}^{\infty} |a^k|^2 = 1 + |a|^2 + |a|^4 + |a|^6 + \cdots \\ |a|^2 S &= |a|^2 + |a|^4 + |a|^6 + \cdots \end{aligned}$$

Subtracting both equations, we have

$$\begin{aligned} (1 - |a|^2)S &= 1 \\ S &= \frac{1}{1 - |a|^2} \end{aligned}$$

Size of a Discrete-Time Signal

Example cont.

If $|a| = 1$, then the summation approaches to ∞ and

$$
\begin{aligned}
P_f &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} |f[k]|^2 \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=0}^{N} |a^k|^2 = \lim_{N\to\infty} \frac{N+1}{2N+1} \\
&= \frac{1}{2} = 0.5.
\end{aligned}
$$

If $|a| > 1$

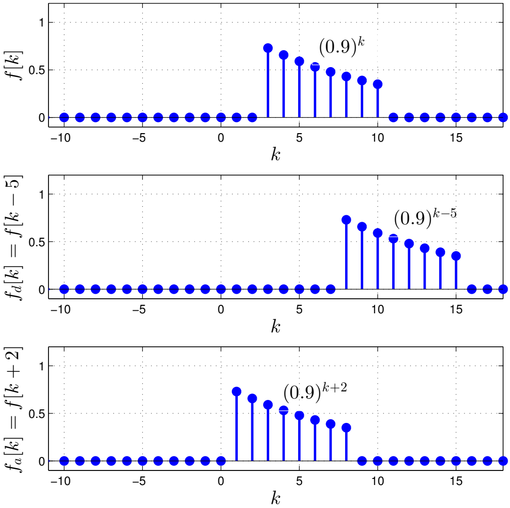
$$
\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=0}^{N} |a^k|^2 = \infty.
$$

To time shift a signal $f[k]$ by m units, we replace k with $k - m$. Thus, $f[k-m]$ represents $f[k]$ time shifted by m units.

- If m is positive, the shift is to the right (delay).
- If m is negative, the shift is to the left (advance).
- Thus $f[k-5]$ is $f[k]$ delayed by 5 units. The signal is the same as $f[k]$ with k replaced by $k-5$. Now, $f[k] = (0.9)^k$ for $3 \leq k \leq 10$. Therefore $f_d[k] = (0.9)^{k-5}$ for $3 \leq k-5 \leq 10$ or $8 \leq k \leq 15$.
- Thus $f[k+2]$ is $f[k]$ advanced by 2 units. The signal is the same as $f[k]$ with k replaced by $k+2$. Now, $f[k] = (0.9)^k$ for $3 \leq k \leq 10$. Therefore $f_a[k] = (0.9)^{k+2}$ for $3 \leq k+2 \leq 10$ or $1 \leq k \leq 8$

Useful Signal Operations

Time Shifting cont.



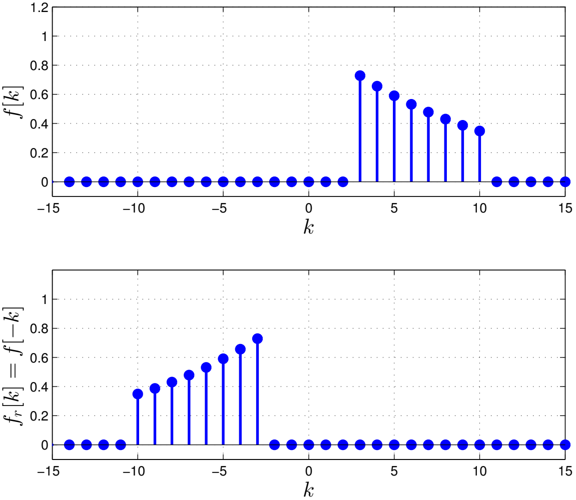
Useful Signal Operations
Time Inversion (or Reversal)

To time invert a signal $f[k]$, we replace k with $-k$. This operation rotates the signal about the vertical axis.

- If $f_r[k]$ is a time-inverted signal $f[k]$, then the expression of $f_r[k]$ is the same as that for $f[k]$ with k replaced by $-k$
- Because $f[k] = (0.9)^k$ for $3 \leq k \leq 10$, $f_r[k] = (0.9)^{-k}$ for $3 \leq -k \leq 10$; that is $-3 \geq k \geq -10$, as shown in Figure on the next slide.

Useful Signal Operations

Time Inversion (or Reversal) cont.



Unlike the continuous-time signal, the discrete-time argument k can take only integer values. Some changes in the procedure are necessary.

Time Compression: Decimation or Downsampling

Consider a signal

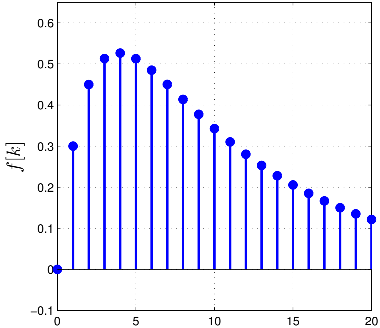
$$f_c[k] = f[2k].$$

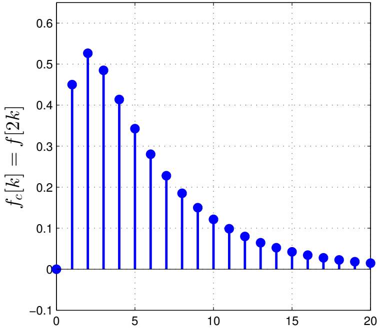
- the signal $f_c[k]$ is the signal $f[k]$ compressed by a factor 2.
- Observe that $f_c[0] = f[0]$, $f_c[1] = f[2]$, $f_c[2] = f[4]$, and so on.
- This fact shows that $f_c[k]$ is made up of even numbered samples of $f[k]$. The odd numbered samples of $f[k]$ are missing.

Useful Signal Operations

Time Compression: Decimation or Downsampling

- This operation loses part of the data, and the time compression is called **decimation** or **downsampling**.
- In general, $f[mk]$ (m integer) consists of only every mth sample of $f[k]$.





Useful Signal Operations

Time Expansion

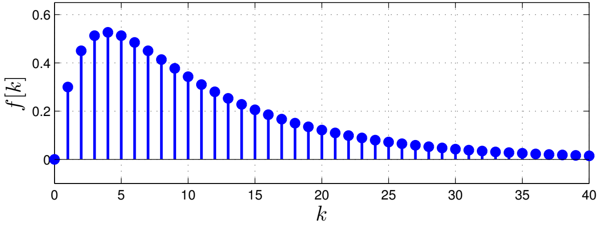
Consider a signal

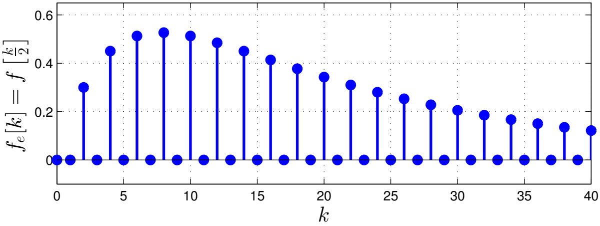
$$f_e[k] = f\left[\frac{k}{2}\right]$$

- the signal $f_e[k]$ is the signal $f[k]$ expanded by a factor 2.
- $f_e[0] = f[0]$, $f_e[1] = f[1/2]$, $f_e[2] = f[1]$, $f_e[3] = f[3/2]$, $f_e[4] = f[2]$, $f_e[5] = f[5/2]$, $f_e[6] = f[3]$, and so on.
- Since, $f[k]$ is defined only for integer values of k, and is zero (or undefined) for all fractional values of k. Therefor for the odd numbered samples $f_e[1]$, $f_e[3]$, $f_e[5]$, $\ldots$ are all zero.
- In general, a function $f_e[k] = f[k/m]$ is defined for $k = 0, \pm m, \pm 2m, \pm 3m, \ldots$ and is zero for all remaining values of k.

Useful Signal Operations

Time Expansion





The missing samples of the discrete-time signal can be reconstructed from the nonzero valued samples using some suitable interpolations formula.

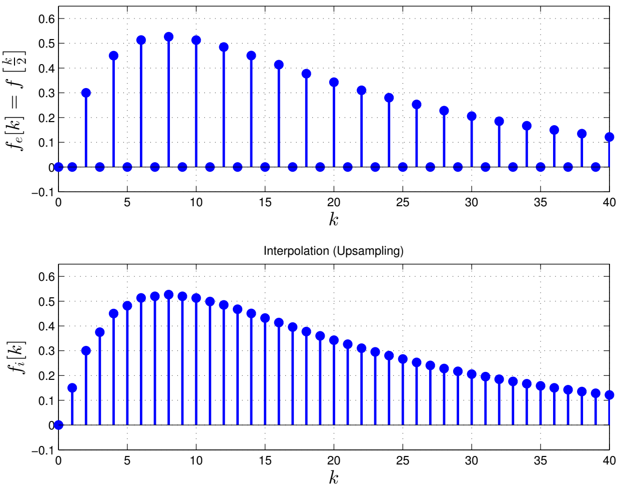
- In practice, we may use a linear interpolation.
- For example

$$f_i[1] \text{ is taken as the mean of } f_i[0] \text{ and } f_i[2]$$
$$f_i[3] \text{ is taken as the mean of } f_i[2] \text{ and } f_i[4], \text{ and so on.}$$

- The process of time expansion and inserting the missing samples using an interpolation is called **interpolation** or **upsampling**.
- In this operation, we increase the number of samples.

Useful Signal Operations

Time Expansion



Examples of Discrete-Time Systems

Example: money deposit

A person makes a deposit (the input) in a bank regularly at an interval of T (say 1 month). The bank pays a certain interest on the account balance during the period T and mails out a periodic statement of the account balance (the output) to the depositor. Find the equation relating the output $y[k]$ (the balance) to the input $f[k]$ (the deposit).

In this case, the signals are inherently discrete-time. Let

$$
\begin{aligned}
f[k] &= \text{ the deposit made at the } k\text{th discrete instant} \\
y[k] &= \text{ the account balance at the } k\text{th instant computed} \\
&\quad\ \text{ immediately after the } k\text{th deposit } f[k] \text{ is received} \\
r &= \text{ interest per dollar per period } T
\end{aligned}
$$

The balance $y[k]$ is the sum of (i) the previous balance $y[k-1]$, (ii) the interest on $y[k-1]$ during the period T, and (iii) the deposit $f[k]$

$$
\begin{aligned}
y[k] &= y[k-1] + ry[k-1] + f[k] \\
&= (1+r)y[k-1] + f[k]
\end{aligned}
$$

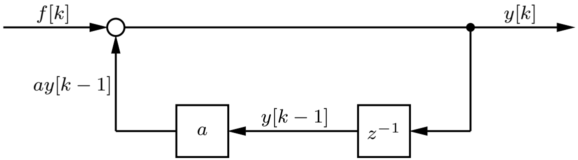
or

$$y[k] - ay[k-1] = f[k] \qquad\qquad a = 1 + r$$

another form

$$y[k+1] - ay[k] = f[k+1]$$

Block diagram or hardware realization



- Assume $y[k]$ is available. Delaying it by one sample, we generate $y[k-1]$.
- We generate $y[k]$ from $f[k]$ and $y[k-1]$

Examples of Discrete-Time Systems

Example: number of students enroll in a course

In the kth semester, $f[k]$ number of students enroll in a course requiring a certain textbook. The publisher sells $y[k]$ new copies of the book in the kth semester. On the average, one quarter of students with book in saleable condition resell their books at the end of semester, and the book life is three semesters. Write the equation relating $y[k]$, the new books sold by the publisher, to $f[k]$, the number of students enrolled in the kth semester, assuming that every student buys a book.

- In the kth semester, the total books $f[k]$ sold to students must be equal to $y[k]$ (new books form the publisher) plus used books from students enrooled in the two previous semesters.
- There are $y[k-1]$ new book sold in the $(k-1)$st semester, and one quarter of these books; that is, $\frac{1}{4}y[k-1]$ will be resold in the kth semester.
- Also $y[k-2]$ new books are sold in the $(k-2)$nd semester, and one quarter of these; that is $\frac{1}{4}y[k-2]$ will be resold in the $(k-1)$st semester.
- Again a quarter of these; that is $\frac{1}{16}y[k-2]$ will be resold in the kth semester. Therefore, $f[k]$ must be equal to the sum of $y[k]$, $\frac{1}{4}y[k-1]$, and $\frac{1}{16}y[k-2]$.

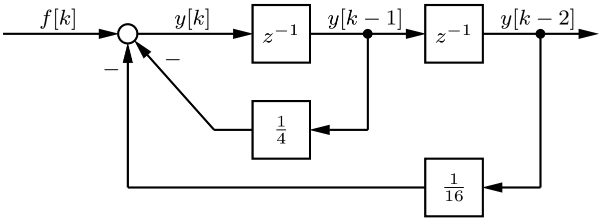
$$y[k] + \frac{1}{4}y[k-1] + \frac{1}{16}y[k-2] = f[k].$$

Above equation can be rewritten as

$$y[k+2] + \frac{1}{4}y[k+1] + \frac{1}{16}y[k] = f[k+2].$$

To make a block diagram the equation is rewritten as

$$y[k] = -\frac{1}{4}y[k-1] - \frac{1}{16}y[k-2] + f[k]$$



Design a discrete-time system to differentiate continuous-time signals.
Since

$$y(t) = \frac{df}{dt}$$

Therefore, at $t = kT$

$$y(kT) = \left.\frac{df}{dt}\right|_{t=kT} = \lim_{T\to 0} \frac{1}{T}\left[f(kT) - f[(k-1)T]\right]$$

By fixing the interval T, the above equation can be expressed as

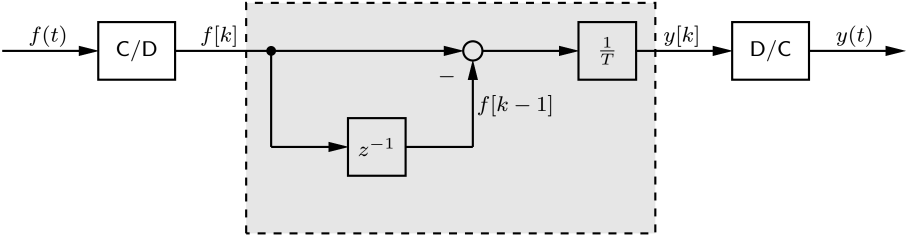
$$y[k] = \lim_{T\to 0} \frac{1}{T}\left\{f[k] - f[k-1]\right\}$$

In practice, the sampling interval T cannot be zero. Assuming T to be sufficiently small, the above equation can be expressed as

$$y[k] \approx \frac{1}{T}\left\{f[k] - f[k-1]\right\}.$$

Examples of Discrete-Time Systems

Example: Discrete-Time Differentiator cont.



Discrete-Time Differentiator Block Diagram

Reference

1. Lathi, B. P., *Signal Processing & Linear Systems*,
Berkeley-Cambridge Press, 1998.