

Lecture 5: Laplace Transform and Its Applications

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Outline

- Motivation
- The Laplace Transform
- The Laplace Transform Properties
- Application to Zero-input and Zero-State Response
- Analysis of Electrical Networks

Motivation

The **Laplace Transform** convert *integral* and *differential* equations into *algebraic* equations.

It can applies to

- general signal, not just sinusoids
- handles transient conditions

It can be used to analyze

- Linear Constant Coefficient Ordinary Differential Equation (LCCODE) or LTI system
- complicated RLC circuits with sources
- complicated systems with integrators, differentiators, gains

The Unilateral Laplace transform

We will be interested in signals defined for $t > 0$.

Definition

Let $f(t), t > 0$, be a given signal (function). The **Unilateral Laplace transform** of a signal (function) $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt,$$

for those $s \in \mathbb{C}$ for which the integral exists.

- F is a complex-valued function of complex numbers
- s is called the *(complex) frequency variable*, with units sec^{-1} ; t is called the *time variable* (in sec); st is unitless.
- For convenience, we will call the unilateral laplace transform as the laplace transform.

The Laplace transform

Example

Exponential function: $f(t) = e^t$

$$F(s) = \int_0^{\infty} e^t e^{-st} dt = \int_0^{\infty} e^{(1-s)t} dt = \frac{1}{1-s} e^{(1-s)t} \Big|_0^{\infty} = \frac{1}{s-1}$$

provide we can say $e^{(1-s)t} \rightarrow 0$ as $t \rightarrow \infty$, which is true for $\operatorname{Re} s > 1$:

$$|e^{(1-s)t}| = \underbrace{|e^{-j(\operatorname{Im} s)t}|}_{=1} |e^{(1-\operatorname{Re} s)t}| = e^{(1-\operatorname{Re} s)t}$$

- the *integral* defining $F(s)$ exists for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. This condition is called **region of convergence (ROC)** of $F(s)$.
- however the resulting *formula* for $F(s)$ makes sense for all $s \in \mathbb{C}$ excepts $s = 1$.

The Laplace transform

Example cont.

Constant or unit step function: $f(t) = u(t)$ (for $t \geq 0$)

$$F(s) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

provided we can say $e^{-st} \rightarrow 0$ as $t \rightarrow \infty$, which is true for $\operatorname{Re} s > 0$ since

$$|e^{-st}| = |e^{-j(\operatorname{Im} s)t}| |e^{-(\operatorname{Re} s)t}| = e^{-(\operatorname{Re} s)t}$$

- the *integral* defining $F(s)$ makes sense for all s with $\operatorname{Re} s > 0$.
- however the resulting *formula* for $F(s)$ makes sense for all s except $s = 0$.

The Laplace transform

Example cont.

Sinusoid : first express $f(t) = \cos \omega t$ as

$$f(t) = \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t}$$

now we can find F as

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \left(\frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} \right) dt \\ &= \frac{1}{2} \int_0^{\infty} e^{(-s+j\omega)t} dt + \frac{1}{2} \int_0^{\infty} e^{(-s-j\omega)t} dt \\ &= \frac{1}{2} \frac{1}{s - j\omega} + \frac{1}{2} \frac{1}{s + j\omega} \\ &= \frac{s}{s^2 + \omega^2} \end{aligned}$$

(valid for $\text{Re } s > 0$; final formula for $s \neq \pm j\omega$)

The Laplace transform

Example cont.

Powers of t : $f(t) = t^n$, ($n \geq 1$)

$$\begin{aligned} F(s) &= \int_0^{\infty} t^n e^{-st} dt = t^n \left(\frac{-e^{-st}}{s} \right) \Big|_0^{\infty} + \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt \\ &= \frac{n}{s} \mathcal{L}(t^{n-1}) \end{aligned}$$

provided $t^n e^{-st} \rightarrow 0$ if $t \rightarrow \infty$, which is true for $\operatorname{Re} s > 0$. Applying the formula recursively, we obtain

$$F(s) = \frac{n!}{s^{n+1}}$$

valid for $\operatorname{Re} s > 0$; final formula exists for all $s \neq 0$.

The Laplace transform

Impulses at $t = 0$

If $f(t)$ contains impulses at $t = 0$ we choose to *include* them in the integral defining $F(s)$:

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

example: impulse function, $f(t) = \delta(t)$

$$F(s) = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^{-st}\Big|_{t=0} = 1 \text{ sampling property}$$

Similarly for $f(t) = \delta^{(k)}(t)$ we have

$$F(s) = \int_{0^-}^{\infty} \delta^{(k)}(t)e^{-st}dt = (-1)^k \frac{d^k}{dt^k} e^{-st}\Big|_{t=0} = s^k e^{-st}\Big|_{t=0} = s^k$$

The Laplace transform

Multiplication by t

Let $f(t)$ be a signal and define

$$g(t) = tf(t) \quad \text{then we have} \quad G(s) = -\frac{d}{ds}F(s)$$

To verify formula, just differentiate both sides of

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

with respect to s to get

$$\begin{aligned} \frac{d}{ds}F(s) &= \int_0^{\infty} (-t)e^{-st} f(t) dt = \int_0^{\infty} (-t)f(t)e^{-st} dt \\ &= - \int_0^{\infty} tf(t)e^{-st} dt = -G(s) \end{aligned}$$

The Laplace transform

Multiplication by t examples

Examples:

- $f(t) = e^{-t}$, $g(t) = te^{-t}$

$$\mathcal{L}\{te^{-t}\} = -\frac{d}{ds} \frac{1}{s+1} = \frac{1}{(s+1)^2}$$

- $f(t) = te^{-t}$, $g(t) = t^2e^{-t}$

$$\mathcal{L}\{t^2e^{-t}\} = -\frac{d}{ds} \frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

- in general

$$\mathcal{L}\{t^k e^{-\lambda t}\} = \frac{k!}{(s+\lambda)^{k+1}}$$

The Laplace transform

Inverse Laplace transform

In principle we can recover $f(t)$ from $F(s)$ via

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$$

where σ is large enough that $F(s)$ is defined for $\operatorname{Re} s \geq \sigma$.

In practical, no one uses this formula!.

Inverse Laplace Transform

Finding the inverse Laplace transform by using the standard formula

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$$

is difficult and tedious.

- We can find the inverse transforms from the transform table.
- All we need is to express $F(s)$ as a sum of simpler functions of the forms listed in the Laplace transform table.
- Most of the transforms $F(s)$ of practical interest are rational functions: that is ratios of polynomials in s .
- Such functions can be expressed as a sum of simpler functions by using partial fraction expansion.

Inverse Laplace Transform

Partial fraction expansion

Example: Find the inverse Laplace transform of $\frac{7s - 6}{s^2 - s - 6}$.

$$F(s) = \frac{7s - 6}{(s + 2)(s - 3)} = \frac{k_1}{s + 2} + \frac{k_2}{s - 3}$$

Using a “cover up” method:

$$k_1 = \left. \frac{7s - 6}{s - 3} \right|_{s=-2} = \frac{-14 - 6}{-2 - 3} = 4$$

$$k_2 = \left. \frac{7s - 6}{s + 2} \right|_{s=3} = \frac{21 - 6}{3 + 2} = 3$$

Therefore

$$F(s) = \frac{7s - 6}{(s + 2)(s - 3)} = \frac{4}{s + 2} + \frac{3}{s - 3}$$

Inverse Laplace Transform

Partial fraction expansion cont.

Using the table of Laplace transforms, we obtain

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \left\{ \frac{4}{s+2} + \frac{3}{s-3} \right\} \\ &= (4e^{-2t} + 3e^{3t}), \quad t \geq 0. \end{aligned}$$

Example: Find the inverse Laplace transform of $F(s) = \frac{2s^2 + 5}{s^2 + 3s + 2}$.

$F(s)$ is an improper function with $m = n$. In such case we can express $F(s)$ as a sum of the coefficient b_n (the coefficient of the highest power in the numerator) plus partial fractions corresponding to the denominator.

$$F(s) = \frac{2s^2 + 5}{(s+1)(s+2)} = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

Inverse Laplace Transform

Partial fraction expansion cont.

where

$$k_1 = \left. \frac{2s^2 + 5}{s + 2} \right|_{s=-1} = \frac{2 + 5}{-1 + 2} = 7$$

and

$$k_2 = \left. \frac{2s^2 + 5}{s + 1} \right|_{s=-2} = \frac{8 + 5}{-2 + 1} = -13$$

Therefore $F(s) = 2 + \frac{7}{s + 1} - \frac{13}{s + 2}$. From the table, we obtain

$$f(t) = 2\delta(t) + 7e^{-t} - 13e^{-2t}, \quad t \geq 0.$$

Inverse Laplace Transform

Partial fraction expansion cont.

Example: Find the inverse Laplace transform of $F(s) = \frac{6(s+34)}{s(s^2+10s+34)}$

$$\begin{aligned} F(s) &= \frac{6(s+34)}{s(s^2+10s+34)} = \frac{6(s+34)}{s(s+5-j3)(s+5+j3)} \\ &= \frac{k_1}{s} + \frac{k_2}{s+5-j3} + \frac{k_2^*}{s+5+j3} \end{aligned}$$

Note that the coefficients (k_2 and k_2^*) of the conjugate terms must also be conjugate. Now

$$\begin{aligned} k_1 &= \left. \frac{6(s+34)}{s^2+10s+34} \right|_{s=0} = \frac{6 \times 34}{34} = 6 \\ k_2 &= \left. \frac{6(s+34)}{s(s+5+j3)} \right|_{s=-5+j3} = \frac{29+j3}{-3-j5} = -3+j4 \\ k_2^* &= -3-j4 \end{aligned}$$

To use the Laplace transform table, we need to express k_2 and k_2^* in polar form

$$-3+j4 = \sqrt{3^2+4^2} e^{j \tan^{-1}(4/-3)} = 5e^{j \tan^{-1}(4/-3)}$$

Inverse Laplace Transform

Partial fraction expansion cont.

From the Figure below, we observe that

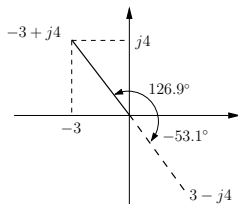
$$k_2 = -3 + j4 = 5e^{j126.9^\circ} \text{ and } k_2^* = 5e^{-j126.9^\circ}$$

Therefore

$$F(s) = \frac{6}{s} + \frac{5e^{j126.9^\circ}}{s + 5 - j3} + \frac{5e^{-j126.9^\circ}}{s + 5 + j3}$$

From the table pair 10b

$$f(t) = [6 + 10e^{-5t} \cos(3t + 126.9^\circ)] u(t)$$



Inverse Laplace Transform

Alternative Method Using Quadratic Factors

$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{k_1}{s} + \frac{As+B}{s^2+10s+34}$$

We have already determined that $k_1 = 6$ by the (Heaviside) “cover-up” method. Therefore

$$\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

Clearing the fractions by multiplying both sides by $s(s^2+10s+34)$ yields

$$\begin{aligned} 6(s+34) &= 6(s^2+10s+34) + s(As+B) \\ &= (6+A)s^2 + (60+B)s + 204 \end{aligned}$$

Now, equating the coefficients of s^2 and s on both sides yields

$$0 = (6+A) \implies A = -6$$

$$6 = 60+B \implies B = -54$$

Inverse Laplace Transform

Alternative Method Using Quadratic Factors cont.

and

$$F(s) = \frac{6}{s} + \frac{-6s - 54}{s^2 + 10s + 34}$$

Now from the table, the parameters for this inverse are $A = -6$, $B = -54$, $a = 5$, $c = 34$, and $b = \sqrt{c - a^2} = 3$, and

$$r = \sqrt{\frac{A^2c + B^2 - 2ABa}{c - a^2}} = 10, \quad \theta = \tan^{-1} \frac{Aa - B}{A\sqrt{c - a^2}} = 126.9^\circ$$
$$b = \sqrt{c - a^2}$$

Therefore

$$f(t) = [6 + 10e^{-5t} \cos(3t + 126.9^\circ)] u(t)$$

which agrees with the previous result.

Inverse Laplace Transform

Alternative Method Using Short-Cuts

$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

This step can be accomplished by multiplying both sides of the above equation by s and then letting $s \rightarrow \infty$. This procedure yields

$$0 = 6 + A \implies A = -6.$$

Therefore

$$\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{-6s+B}{s^2+10s+34}$$

To find B , we let s take on any convenient value, say $s = 1$, in this equation to obtain

$$\frac{210}{45} = 6 + \frac{B-6}{45}$$

Inverse Laplace Transform

Alternative Method Using Short-Cuts cont.

Multiplying both sides of this equation by 45 yields

$$210 = 270 + B - 6 \implies B = -54$$

a deduction which agrees with the results we found earlier.

Inverse Laplace Transform

Partial fraction expansion: repeated roots

Example: Find the inverse Laplace transform of $F(s) = \frac{8s + 10}{(s + 1)(s + 2)^3}$

$$F(s) = \frac{8s + 10}{(s + 1)(s + 2)^3} = \frac{k_1}{s + 1} + \frac{a_0}{(s + 2)^3} + \frac{a_1}{(s + 2)^2} + \frac{a_2}{s + 2}$$

where

$$k_1 = \left. \frac{8s + 10}{(s + 2)^3} \right|_{s=-1} = 2$$

$$a_0 = \left. \frac{8s + 10}{(s + 1)} \right|_{s=-2} = 6$$

$$a_1 = \left\{ \frac{d}{ds} \left[\frac{8s + 10}{(s + 1)} \right] \right\}_{s=-2} = -2$$

$$a_2 = \frac{1}{2} \left\{ \frac{d^2}{ds^2} \left[\frac{8s + 10}{(s + 1)} \right] \right\}_{s=-2} = -2$$

Note : the general formula is

$$a_n = \frac{1}{n!} \left\{ \frac{d^n}{ds^n} [(s - \lambda)^r F(s)] \right\}_{s=\lambda}$$

Inverse Laplace Transform

Partial fraction expansion: repeated roots

Therefore

$$F(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$$

and

$$f(t) = [2e^{-t} + (3t^2 - 2t - 2)e^{-2t}] u(t)$$

Alternative Method: A Hybrid of Heaviside and Clearing Fractions: Using the values $k_1 = 2$ and $a_0 = 6$ obtained earlier by the Heaviside “cover-up” method, we have

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2}$$

Inverse Laplace Transform

Partial fraction expansion: repeated roots

We now clear fractions by multiplying both sides of the equation by $(s+1)(s+2)^3$. This procedure yields

$$\begin{aligned}8s + 10 &= 2(s+2)^3 + 6(s+1) + a_1(s+1)(s+2) + a_2(s+1)(s+2)^2 \\&= (2+a_2)s^3 + (12+a_1+5a_2)s^2 + (30+3a_1+8a_2)s + (22+2a_1+4a_2)\end{aligned}$$

Equating coefficients of s^3 and s^2 on both sides, we obtain

$$0 = (2 + a_2) \implies a_2 = -2$$

$$0 = 12 + a_1 + 5a_2 = 2 + a_1 \implies a_1 = -2$$

Equating the coefficients of s^1 and s^0 serves as a check on our answers.

$$8 = 30 + 3a_1 + 8a_2$$

$$10 = 22 + 2a_1 + 4a_2$$

Substitution of $a_1 = a_2 = -2$, obtained earlier, satisfies these equations.

Inverse Laplace Transform

Partial fraction expansion: repeated roots

Alternative Method: A Hybrid of Heaviside and Short-Cuts: Using the values $k_1 = 2$ and $a_0 = 6$, determined earlier by the Heaviside method, we have

$$\frac{8s + 10}{(s + 1)(s + 2)^3} = \frac{2}{s + 1} + \frac{6}{(s + 2)^3} + \frac{a_1}{(s + 2)^2} + \frac{a_2}{s + 2}$$

There are two unknowns, a_1 and a_2 . If we multiply both sides by s and then let $s \rightarrow \infty$, we eliminate a_1 . This procedure yields

$$0 = 2 + a_2 \implies a_2 = -2$$

Therefore

$$\frac{8s + 10}{(s + 1)(s + 2)^3} = \frac{2}{s + 1} + \frac{6}{(s + 2)^3} + \frac{a_1}{(s + 2)^2} - \frac{2}{s + 2}$$

There is now only one unknown, a_1 . This value can be determined readily by equal to any convenient value, say $s = 0$. This step yields

$$\frac{10}{8} = 2 + \frac{3}{4} + \frac{a_1}{4} - 1 \implies a_1 = -2.$$

The Laplace transform properties

Linearity

The Laplace transform is *linear*: if $f(t)$ and $g(t)$ are any signals, and a is any scalar, we have

$$\mathcal{L}\{af(t)\} = aF(s), \quad \mathcal{L}\{(f(t) + g(t))\} = F(s) + G(s)$$

i.e., homogeneity and superposition hold.

Example:

$$\begin{aligned}\mathcal{L}\{3\delta(t) - 2e^t\} &= 3\mathcal{L}\{\delta(t)\} - 2\mathcal{L}\{e^t\} \\ &= 3 - \frac{2}{s-1} \\ &= \frac{3s-5}{s-1}\end{aligned}$$

The Laplace transform properties

One-to-one property

The Laplace transform is *one-to-one*: if $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ then $f(t) = g(t)$.

- $F(s)$ determines $f(t)$
- inverse Laplace transform $\mathcal{L}^{-1}\{f(t)\}$ is well defined.

Example:

$$\mathcal{L}^{-1}\left\{\frac{3s-5}{s-1}\right\} = 3\delta(t) - 2e^t$$

in other words, the only function $f(t)$ such that

$$F(s) = \frac{3s-5}{s-1}$$

is $f(t) = 3\delta(t) - 2e^t$.

The Laplace transform properties

Time delay

This property states that if

$$f(t) \Longleftrightarrow F(s)$$

then for $T \geq 0$

$$f(t - T) \Longleftrightarrow e^{-sT} F(s)$$

(If $g(t)$ is $f(t)$, delayed by T seconds), then we have $G(s) = e^{-sT} F(s)$.

Derivation:

$$\begin{aligned} G(s) &= \int_0^{\infty} e^{-st} g(t) dt = \int_0^{\infty} e^{-st} f(t - T) dt \\ &= \int_0^{\infty} e^{-s(\tau+T)} f(\tau) d\tau = e^{-sT} F(s) \end{aligned}$$

The Laplace transform properties

Time delay

To avoid a pitfall, we should restate the property as follow:

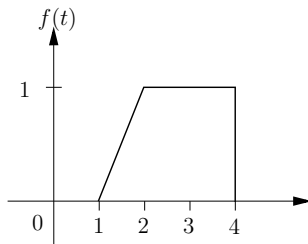
$$f(t)u(t) \iff F(s)$$

then

$$f(t - T)u(t - T) \iff e^{-sT}F(s), \quad T \geq 0.$$

The Laplace transform properties

Time delay example



Find the Laplace Transform of $f(t)$ depicted in Figure above.

The signal can be described as

$$\begin{aligned}f(t) &= (t-1)[u(t-1) - u(t-2)] + [u(t-2) - u(t-4)] \\&= (t-1)u(t-1) - (t-1)u(t-2) + u(t-2) - u(t-4) \\&= (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)\end{aligned}$$

The Laplace transform properties

Time delay example

Since $t \iff \frac{1}{s^2}$ yields

$$(t-1)u(t-1) \iff \frac{1}{s^2}e^{-s} \text{ and } (t-2)u(t-2) \iff \frac{1}{s^2}e^{-2s}$$

Also $u(t) \iff \frac{1}{s}$ yields

$$u(t-4) \iff \frac{1}{s}e^{-4s}$$

Therefore

$$F(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

The Laplace transform properties

Time delay example

Find the inverse Laplace transform of

$$F(s) = \frac{s + 3 + 5e^{-2s}}{(s + 1)(s + 2)}$$

The $F(s)$ can be separated in two parts

$$F(s) = \underbrace{\frac{s + 3}{(s + 1)(s + 2)}}_{F_1(s)} + \underbrace{\frac{5e^{-2s}}{(s + 1)(s + 2)}}_{F_2(s)e^{-2s}}$$

where

$$F_1(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{2}{s + 1} - \frac{1}{s + 2}$$
$$F_2(s) = \frac{5}{(s + 1)(s + 2)} = \frac{5}{s + 1} - \frac{5}{s + 2}$$

The Laplace transform properties

Time delay example

Therefore

$$f_1(t) = (2e^{-t} - e^{-2t})$$

$$f_2(t) = 5(e^{-t} - e^{-2t})$$

Since

$$F(s) = F_1(s) + F_2(s)e^{-2s}$$

$$f(t) = f_1(t) + f_2(t - 2)$$

$$= (2e^{-t} - e^{-2t}) u(t) + 5 \left[e^{-(t-2)} - e^{-2(t-2)} \right] u(t - 2)$$

The Laplace transform properties

Time scaling

Define a signal $g(t)$ by $g(t) = f(at)$, where $a > 0$; then

$$G(s) = \frac{1}{a} F\left(\frac{s}{a}\right).$$

time are scaled by a , then frequencies are scaled by $1/a$.

$$G(s) = \int_0^{\infty} f(at)e^{-st}dt = \frac{1}{a} \int_0^{\infty} f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a} F\left(\frac{s}{a}\right),$$

where $\tau = at$.

Example: $\mathcal{L}\{e^t\} = \frac{1}{s-1}$ so

$$\mathcal{L}\{e^{at}\} = \frac{1}{a} \frac{1}{\frac{s}{a} - 1} = \frac{1}{s-a}$$

The Laplace transform properties

Exponential scaling

Let $f(t)$ be a signal and a a scale, and define $g(t) = e^{at}f(t)$; then

$$G(s) = F(s - a)$$

Proof:

$$G(s) = \int_0^{\infty} e^{-st} e^{at} f(t) dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt = F(s - a)$$

Example: $\mathcal{L}\{\cos t\} = \frac{s}{s^2 + 1}$, and hence

$$\mathcal{L}\{e^{-t} \cos t\} = \frac{s + 1}{(s + 1)^2 + 1} = \frac{s + 1}{s^2 + 2s + 2}$$

The Laplace transform properties

Exponential scaling

Example: Consider $F(s) = \frac{-6s - 54}{s^2 + 10s + 34}$. By using the exponential exponential scaling, we obtain

$$\frac{-6s - 54}{s^2 + 10s + 34} = \frac{-6(s + 5) - 24}{(s + 5)^2 + 9} = \frac{-6(s + 5)}{(s + 5)^2 + 3^2} + \frac{-8(3)}{(s + 5)^2 + 3^2}$$

Then,

$$\begin{aligned} f(t) &= -6e^{-5t} \cos 3t - 8e^{-5t} \sin 3t \\ &= 10e^{-5t} \cos(3t + 127^\circ) \end{aligned}$$

You can do this inverse Laplace transform using only standard Laplace transform table.

The Laplace transform properties

Derivative

If signal $f(t)$ is continuous at $t = 0$, then

$$\mathcal{L} \left\{ \frac{df}{dt} \right\} = sF(s) - f(0);$$

- time-domain differentiation becomes multiplication by frequency variable s (as with phasors)
- plus a term that includes initial condition (i.e., $-f(0)$)

higher-order derivatives: applying derivative formula twice yields

$$\begin{aligned} \mathcal{L} \left\{ \frac{d^2 f(t)}{dt^2} \right\} &= s \mathcal{L} \left\{ \frac{df(t)}{dt} \right\} - \frac{df(t)}{dt} \\ &= s(sF(s) - f(0)) - \frac{df(0)}{dt} = s^2 F(s) - sf(0) - \frac{df(0)}{dt} \end{aligned}$$

similar formulas hold for $\mathcal{L} \{ f^{(k)} \}$.

The Laplace transform properties

Derivation of derivative formula

Start from the defining integral

$$G(s) = \int_0^{\infty} \frac{df(t)}{dt} e^{-st} dt$$

integration by parts yields

$$\begin{aligned} G(s) &= e^{-st} f(t) \Big|_0^{\infty} - \int_0^{\infty} f(t) (-se^{-st}) dt \\ &= f(t) e^{-s\infty} - f(0) + sF(s) \end{aligned}$$

we recover the formula

$$G(s) = sF(s) - f(0)$$

The Laplace transform properties

Derivative example

1. $f(t) = e^t$, so $f'(t) = e^t$ and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{f'(t)\} = \frac{1}{s-1}$$

by using $\mathcal{L}\{f'(t)\} = s \frac{1}{s-1} - 1$, which is the same.

2. $\sin \omega t = -\frac{1}{\omega} \frac{d}{dt} \cos \omega t$, so

$$\mathcal{L}\{\sin \omega t\} = -\frac{1}{\omega} \left(s \frac{s}{s^2 + \omega^2} - 1 \right) = \frac{\omega}{s^2 + \omega^2}$$

3. $f(t)$ is a unit ramp, so $f'(t)$ is a unit step

$$\mathcal{L}\{f'(t)\} = s \left(\frac{1}{s^2} \right) - 0 = \frac{1}{s}$$

The Laplace transform properties

Integral

Let $g(t)$ be the running integral of a signal $f(t)$, i.e.,

$$g(t) = \int_0^t f(\tau) d\tau$$

then $G(s) = \frac{1}{s} F(s)$, i.e., *time-domain integral* become division by frequency variable s .

Example: $f(t) = \delta(t)$ is a unit impulse function, so $F(s) = 1$; $g(t)$ is the unit step

$$G(s) = \frac{1}{s}.$$

Example: $f(t)$ is a unit step function, so $F(s) = 1/s$; $g(t)$ is the unit ramp function ($g(t) = t$ for $t \geq 0$),

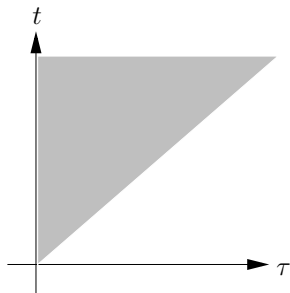
$$G(s) = \frac{1}{s^2}$$

The Laplace transform properties

Derivation of integral formula:

$$G(s) = \int_{t=0}^{\infty} \left(\int_{\tau=0}^t f(\tau) d\tau \right) e^{-st} dt$$

here we integrate horizontally first over the triangle $0 \leq \tau \leq t$.



Let's switch the order, integrate vertically first:

$$\begin{aligned} G(s) &= \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} f(\tau) e^{-st} dt d\tau \\ &= \int_{\tau=0}^{\infty} f(\tau) \left(\int_{t=\tau}^{\infty} e^{-st} dt \right) d\tau \\ &= \int_{\tau=0}^{\infty} f(\tau) \frac{1}{s} e^{-s\tau} d\tau = \frac{F(s)}{s} \end{aligned}$$

The Laplace transform properties

Convolution

The convolution of signals $f(t)$ and $g(t)$, denoted $h(t) = f(t) * g(t)$, is the signal

$$h(t) = \int_0^t f(\tau)g(t - \tau)d\tau$$

In terms of Laplace transforms:

$$H(s) = F(s)G(s)$$

The Laplace transform turns convolution into multiplication.

The Laplace transform properties

Convolution cont.

Let's show that $\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$:

$$\begin{aligned} H(s) &= \int_{t=0}^{\infty} e^{-st} \left(\int_{\tau=0}^t f(\tau)g(t-\tau)d\tau \right) dt \\ &= \int_{t=0}^{\infty} \int_{\tau=0}^t e^{-st} f(\tau)g(t-\tau)d\tau dt \end{aligned}$$

where we integrate over the triangle $0 \leq \tau \leq t$. By changing the order of the integration and changing the limits of integration yield

$$H(s) = \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} e^{-st} f(\tau)g(t-\tau)dt d\tau$$

The Laplace transform properties

Convolution cont.

Change variable t to $\bar{t} = t - \tau$; $d\bar{t} = dt$; region of integration becomes $\tau \geq 0, \bar{t} \geq 0$

$$\begin{aligned} H(s) &= \int_{\tau=0}^{\infty} \int_{\bar{t}=0}^{\infty} e^{-s(\bar{t}+\tau)} f(\tau) g(\bar{t}) d\bar{t} d\tau \\ &= \left(\int_{\tau=0}^{\infty} e^{-s\tau} f(\tau) d\tau \right) \left(\int_{\bar{t}=0}^{\infty} e^{-s\bar{t}} g(\bar{t}) d\bar{t} \right) \\ &= F(s)G(s) \end{aligned}$$

The Laplace transform properties

Convolution cont.

Example: Using the time convolution property of the Laplace transform, determine $c(t) = e^{at}u(t) * e^{bt}u(t)$. From the convolution property, we have

$$C(s) = \frac{1}{s-a} \frac{1}{s-b} = \frac{1}{a-b} \left[\frac{1}{s-a} - \frac{1}{s-b} \right]$$

The inverse transform of the above equation yields

$$c(t) = \frac{1}{a-b} (e^{at} - e^{bt}), \quad t \geq 0.$$

Applications

Solution of Differential and Integro-Differential Equations

Solve the second-order linear differential equation

$$(D^2 + 5D + 6)y(t) = (D + 1)f(t)$$

if the initial conditions are $y(0^-) = 2$, $\dot{y}(0^-) = 1$, and the input $f(t) = e^{-4t}u(t)$.

The equation is

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{df}{dt} + f(t).$$

Let

$$y(t) \Longleftrightarrow Y(s).$$

Then

$$\frac{dy}{dt} \Longleftrightarrow sY(s) - y(0^-) = sY(s) - 2.$$

Applications

Solution of Differential and Integro-Differential Equations

and

$$\frac{d^2 y}{dt^2} \Longleftrightarrow s^2 Y(s) - sy(0^-) - \dot{y}(0^-) = s^2 Y(s) - 2s - 1.$$

Moreover, for $f(t) = e^{-4t}u(t)$,

$$F(s) = \frac{1}{s+4}, \text{ and } \frac{df}{dt} \Longleftrightarrow sF(s) - f(0^-) = \frac{s}{s+4} - 0 = \frac{s}{s+4}.$$

Taking the Laplace transform, we obtain

$$[s^2 Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = \frac{s}{s+4} + \frac{1}{s+4}$$

Collecting all the terms of $Y(s)$ and the remaining terms separately on the left-hand side, we obtain

$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s+1}{s+4}$$

Applications

Solution of Differential and Integro-Differential Equations

Therefore

$$(s^2 + 5s + 6)Y(s) = (2s + 11) + \frac{s + 1}{s + 4} = \frac{2s^2 + 20s + 45}{s + 4}$$

and

$$\begin{aligned} Y(s) &= \frac{2s^2 + 20s + 45}{(s^2 + 5s + 6)(s + 4)} \\ &= \frac{2s^2 + 20s + 45}{(s + 2)(s + 3)(s + 4)} \end{aligned}$$

Expanding the right-hand side into partial fractions yields

$$Y(s) = \frac{13/2}{s + 2} - \frac{3}{s + 3} - \frac{3/2}{s + 4}$$

The inverse Laplace transform of the above equation yields

$$y(t) = \left(\frac{13}{2}e^{-2t} - 3e^{-3t} - \frac{3}{2}e^{-4t} \right) u(t).$$

Applications

Zero-Input and Zero-State Components of Response

- The Laplace transform method gives the total response, which includes zero-input and zero-state components.
- The initial condition terms in the response give rise to the zero-input response.

For example in the previous example,

$$(s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s + 1}{s + 4}$$

so that

$$(s^2 + 5s + 6)Y(s) = \underbrace{(2s + 11)}_{\text{initial condition terms}} + \underbrace{\frac{s + 1}{s + 4}}_{\text{input terms}}$$

Applications

Zero-Input and Zero-State Components of Response

Therefore

$$\begin{aligned} Y(s) &= \underbrace{\frac{2s + 11}{s^2 + 5s + 6}}_{\text{zero-input component}} + \underbrace{\frac{s + 1}{(s + 4)(s^2 + 5s + 6)}}_{\text{zero-state component}} \\ &= \left[\frac{7}{s + 2} - \frac{5}{s + 3} \right] + \left[\frac{-1/2}{s + 2} + \frac{2}{s + 3} - \frac{3/2}{s + 4} \right] \end{aligned}$$

Taking the inverse transform of this equation yields

$$y(t) = \underbrace{(7e^{-2t} - 5e^{-3t})u(t)}_{\text{zero-input response}} + \underbrace{\left(-\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-4t}\right)u(t)}_{\text{zero-state response}}$$

Analysis of Electrical Networks

Basic concept

- It is possible to analyze electrical networks directly without having to write the integro-differential equation.
- This procedure is considerably simpler because it permits us to treat an electrical network as if it was a resistive network.
- To do such a procedure, we need to represent a network in “frequency domain” where all the voltages and currents are represented by their Laplace transforms.

Analysis of Electrical Networks

Basic concept

zero initial conditions case:

If $v(t)$ and $i(t)$ are the voltage across and the current through an inductor of L henries, then

$$v(t) = L \frac{di(t)}{dt} \iff V(s) = sLI(s), \quad i(0) = 0.$$

Similarly, for a capacitor of C farads, the voltage-current relationship is

$$i(t) = C \frac{dv(t)}{dt} \iff V(s) = \frac{1}{Cs} I(s), \quad v(0) = 0.$$

For a resistor of R ohms, the voltage-current relationship is

$$v(t) = Ri(t) \iff V(s) = RI(s).$$

Analysis of Electrical Networks

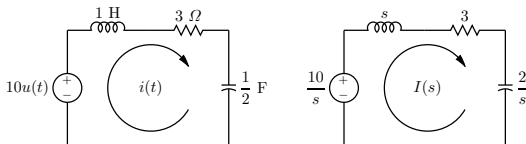
Basic concept

- Thus, in the “frequency domain,” the voltage-current relationships of an inductor and a capacitor are algebraic;
- These elements behave like resistors of “resistance” Ls and $1/Cs$, respectively.
- The generalized “resistance” of an element is called its **impedance** and is given by the ratio $V(s)/I(s)$ for the element (under zero initial conditions).
- The impedances of a resistor of R ohms, and inductor of L henries, and a capacitance of C farads are R , Ls , and $1/Cs$, respectively.
- The Kirchhoff’s laws remain valid for voltages and currents in the frequency domain.

Analysis of Electrical Networks

A simple RC circuit

Find the loop current $i(t)$ in the circuit, if all the initial conditions are zero.



In the first step, we represent the circuit in the frequency domain shown in the right hand side. The impedance in the loop is

$$Z(s) = s + 3 + \frac{2}{s} = \frac{s^2 + 3s + 2}{s}$$

The input voltage is $V(s) = 10/s$. Therefore, the loop current $I(s)$ is

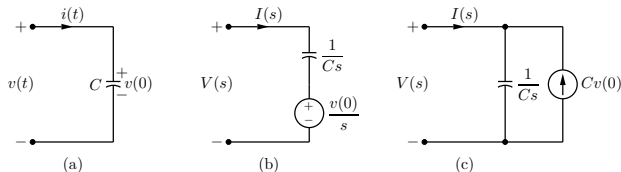
$$I(s) = \frac{V(s)}{Z(s)} = \frac{10/s}{(s^2 + 3s + 2)/s} = \frac{10}{s^2 + 3s + 2} = \frac{10}{(s+1)(s+2)} = \frac{10}{s+1} - \frac{10}{s+2}$$

The inverse transform of the equation yields: $i(t) = 10(e^{-t} - e^{-2t})u(t)$.

Analysis of Electrical Networks

Initial Condition Generators

A capacitor C with an initial voltage $v(0)$ can be represented in the frequency domain by an uncharged capacitor of impedance $1/Cs$ in series with a voltage source of value $v(0)/s$ or as the same uncharged capacitor in parallel with a current source of value $Cv(0)$.



$$i(t) = C \frac{dv}{dt} \iff I(s) = C[sV(s) - v(0)]$$

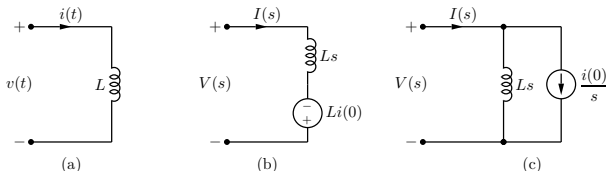
Rearranging the equation, we obtain

$$V(s) = \frac{1}{Cs} I(s) + \frac{v(0)}{s} \text{ or } V(s) = \frac{1}{Cs} [I(s) + Cv(0)]$$

Analysis of Electrical Networks

Initial Condition Generators

An inductor L with an initial voltage $i(0)$ can be represented in the frequency domain by an inductor of impedance Ls in series with a voltage source of value $Li(0)$ or by the same inductor in parallel with a current source of value $i(0)/s$.



$$v(t) = L \frac{di}{dt} \iff V(s) = L[sI(s) - i(0)]$$

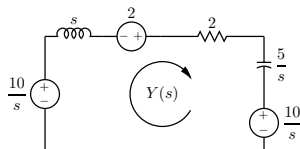
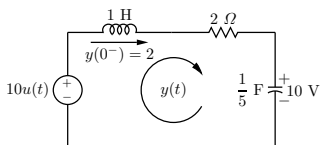
Rearranging the equation, we obtain

$$V(s) = sLI(s) - Li(0) \text{ or } V(s) = Ls \left[I(s) - \frac{i(0)}{s} \right]$$

Analysis of Electrical Networks

A simple RLC circuit with initial condition generators

Find the loop current $i(t)$ in the circuit, if $y(0) = 2$ and $v_C(0) = 10$.



The right hand side figure shows the frequency-domain representation of the circuit. Applying mesh analysis we have

$$\begin{aligned} -\frac{10}{s} + sY(s) - 2 + 2Y(s) + \frac{5}{s}Y(s) + \frac{10}{s} &= 0 \\ Y(s) &= \frac{2}{s + 2 + \frac{5}{s}} \\ &= \frac{2s}{s^2 + 2s + 5} \end{aligned}$$

Analysis of Electrical Networks

A simple RLC circuit with initial condition generators

$$Y(s) = \frac{2s}{s^2 + 2s + 5} = \frac{2(s+1)}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2}$$

Therefore

$$y(t) = e^{-t}(2 \cos 2t - \sin 2t) = e^{-t}(C \cos \theta \cos 2t - C \sin \theta \sin 2t),$$

since

$$C = \sqrt{2^2 + 1} = \sqrt{5}, \quad \theta = \tan^{-1} \frac{2}{4} = 26.6^\circ$$

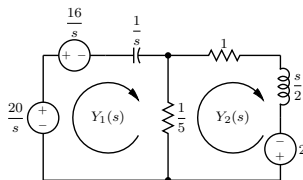
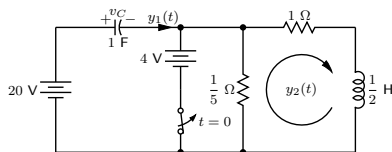
then

$$y(t) = \sqrt{5}e^{-t} \cos(2t + 26.6^\circ)u(t).$$

Analysis of Electrical Networks

An RLC circuit with initial condition generators

The switch in the circuit is in the closed position for a long time before $t = 0$, when it is opened instantaneously. Find the currents $y_1(t)$ and $y_2(t)$ for $t \geq 0$.



When the switch is closed and the steady-state conditions are reached, the capacitor voltage $v_C = 16$ volts, and the inductor current $y_2 = 4$ A. The right hand side circuit shows the transformed version of the circuit in the left hand side. Using mesh analysis, we obtain

$$\begin{aligned}\frac{Y_1(s)}{s} + \frac{1}{5} [Y_1(s) - Y_2(s)] &= \frac{4}{s} \\ -\frac{1}{5} Y_1(s) + \frac{6}{5} Y_2(s) + \frac{s}{2} Y_2(s) &= 2\end{aligned}$$

Analysis of Electrical Networks

An RLC circuit with initial condition generators

Rewriting in matrix form, we have

$$\begin{bmatrix} \frac{1}{s} + \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{6}{5} + \frac{s}{2} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{4}{s} \\ 2 \end{bmatrix}$$

Therefore,

$$\begin{aligned} Y_1(s) &= \frac{24(s+2)}{s^2 + 7s + 12} \\ &= \frac{24(s+2)}{(s+3)(s+4)} = \frac{-24}{s+3} + \frac{48}{s+4} \\ Y_2(s) &= \frac{4(s+7)}{s^2 + 7s + 12} = \frac{16}{s+3} - \frac{12}{s+4}. \end{aligned}$$

Finally,

$$\begin{aligned} y_1(t) &= (-24e^{-3t} + 48e^{-4t})u(t) \\ y_2(t) &= (16e^{-3t} - 12e^{-4t})u(t) \end{aligned}$$

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