Lecture 0: Background

Dr.-Ing. Sudchai Boonto

Assistant Professor

Department of Control System and Instrumentation Engineering
King Mongkut's Unniversity of Technology Thonburi
Thailand

Outline

- The basic for the rest of the material in this course.

Signals and Systems and Digital Technologies

- In modern world, signals of all kinds emanate from different types of devices–radios and TVs, cell phones, global positioning systems (GPSs), radars, and sonar etc.
- These systems allow us to communicate messages, to control processed, and to sense or measure signals.
- The advent of the transistor, the digital computer, and the theoretical fundamentals of digital signal processing, the trend has been toward digital representation and processing of data, most of which are in analog form.
- Such a tend highlights the importance of learning how to represent signals in analog as well as in digital forms and how to model and design systems capable of dealing with different types of signals.

Signal Transmission

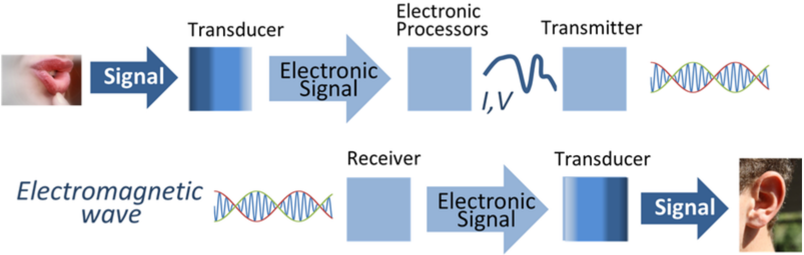
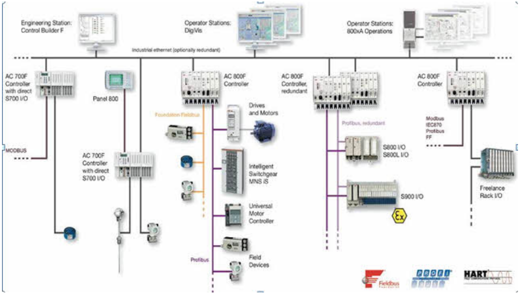


Figure: Signal transmission using electronic signals (from wikipedia)

Computer-Controlled Systems

(a)



(b)

Figure: Computer-controlled system

Examples of Signal Processing Applications

Figure: Wireless Instrument

Examples of Signal Processing Applications

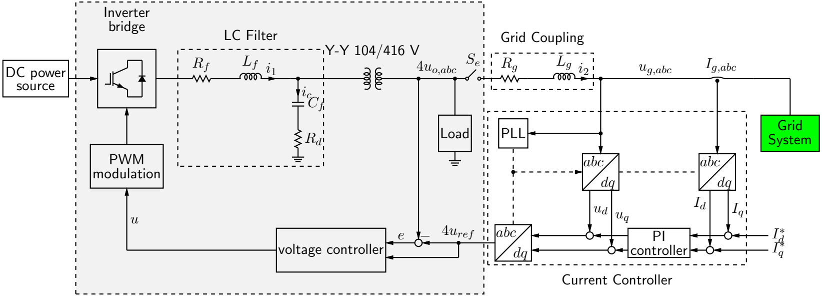


Figure: Grid-Connected System

Examples of Signal Processing Applications

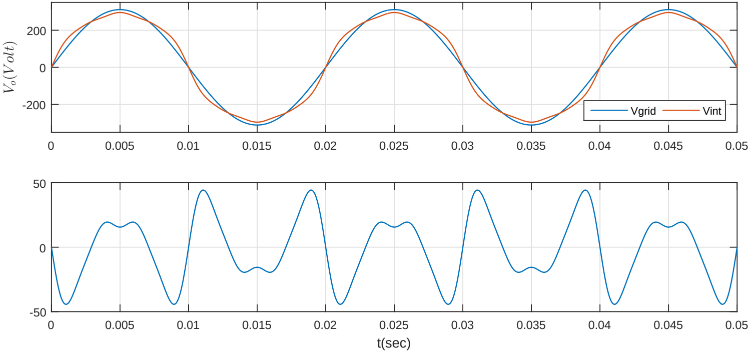


Figure: Grid-Connected System

Introduction to Signals

- Dimension and units of signals
- Classification of signals
- Continuous-Time and Discrete-Time Representation

Dimension and units of a signal

Dimension or **type** of a signal $u(t)$, e.g.,

- **real-valued** or **scalar signal**: $u(t)$ is a real number (scalar)
- **vector signal**: $u(t)$ is a vector of some dimensions
- **binary signal**: $u(t)$ is either 0 or 1

we will usually encounter scalar signals.

example: a vector-valued signal

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

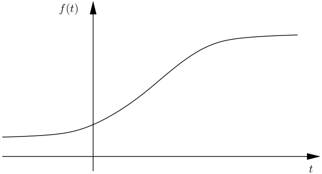
might give the voltage at three places on an antenna.

physical units of a signal, e.g. V, mA, m/sec

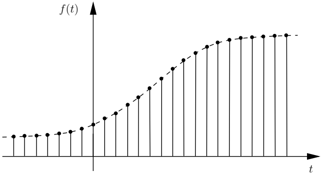
Classification of Signals

- an **Analog signal** is a signal whose amplitude can take on any value in a continuous range
- a **continuous-time signal** is a signal that is specified for every value of time t
- a **discrete-time signal** is a signal that is specified only discrete values of t
- a **digital signal** is a signal whose amplitude can take on only a finite number of values

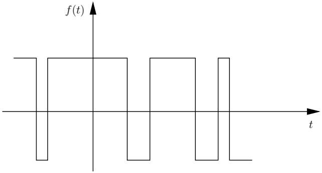
Classification of Signals



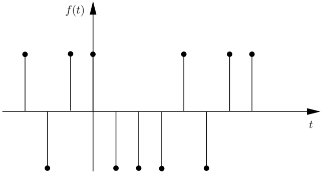
(a) analog, continuous-time signal



(b) analog, discrete-time signal



(c) digital, continuous-time signal



(d) digital, discrete-time signal

- a **periodic signal:** a signal $f(t)$ is said to be periodic if for some positive constant T_0

$$f(t) = f(t + T_0) \qquad \text{for all } t$$

- a signal is **aperiodic** if it is not periodic

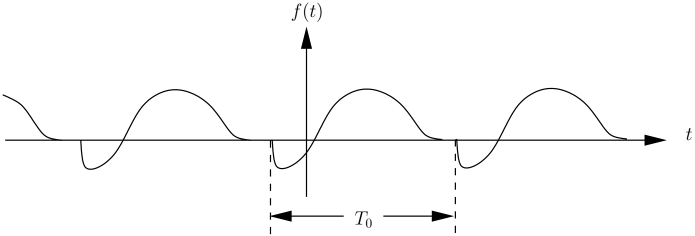


Figure: a periodic signal of period T_0

Classification of Signals

Deterministic and Random Signals

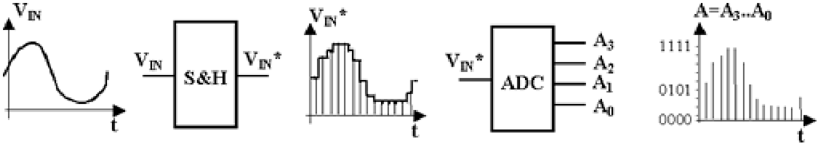
- a **Deterministic signal** is a signal whose physical description is known completely, either in a mathematical form or a graphical form.
- a **Random signal** is a signal whose values cannot be predicted precisely but are known only in terms of probabilistic description, such as mean value, mean squared value, and so on.
- In this course, we deal only with deterministic signals.

Continuous or Discrete?

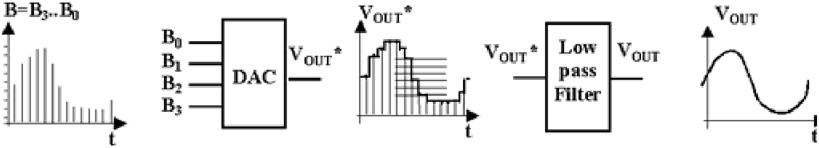
- In engineering, the inputs and outputs of electrical, mechanical, chemical, and biological processes and measured as functions of time with amplitudes expressed in terms of voltage, current, torque, pressure, etc.
- These functions are called *analog* and *continuous-time signals*, and to process them with a computer they must be converted into binary number, 0 or 1.
- A conversion is done in a way as to preserve as much as possible the information contained in the original signal.

Continuous of Discrete?

Analog to Digital Converter converts an analog input to a digital output



Digital to Analog Converter converts a digital signal to an analog output



Continuous or Discrete?

- In binary form, signals can be processed using algorithms (coded procedures understood by computers and designed to obtain certain desired information from the signals or to change them) in a computer or in a dedicated piece of hardware.
- In analog world, we use *calculus* deals with functions of one or more continuously changing variables. The concepts of *derivative* and *integral* are developed to measure the rate of change of functions and the areas under the graphs of these functions, or their volumes. Differential equations are then introduced to characterize dynamic systems.

Continuous or Discrete?

- In discrete world, we use *finite calculus* to deal with sequences number. Thus derivative and integrals are replaced by differences and summations, while differential equations are replaced by difference equations.
- Finite calculus makes possible the computations of calculus by means of a combination of digital computers and numerical methods–thus, finite calculus becomes the more concrete mathematics. Numerical methods applied to sequences permit us to approximate derivatives, integrals, and the solution of differential equations.

Continuous-Time and Discrete-Time Representations

- the analog signal depends continuously on time.
- A discrete-time signal is a sequence of measurements typically made at uniform times, while the analog signal depends continuously on time.
- a discrete-time signal $x[n]$ and the corresponding analog signal $x(t)$ are related by a sampling process:

$$x[n] = x(nT_s) = x(t)|_{t=nT_s}$$

Continuous-Time and Discrete-Time Representations

- the signal $x[n]$ is obtained by sampling $x(t)$ at time $t = nT_s$, where n is an integer and T_s is the *sampling period* or the time between samples. This results in a sequence,

$$\{ \quad \cdots \quad x(-T_s) \quad x(0) \quad x(T_s) \quad x(2T_2) \quad \cdots \quad \}$$

- the sequence number can be written as

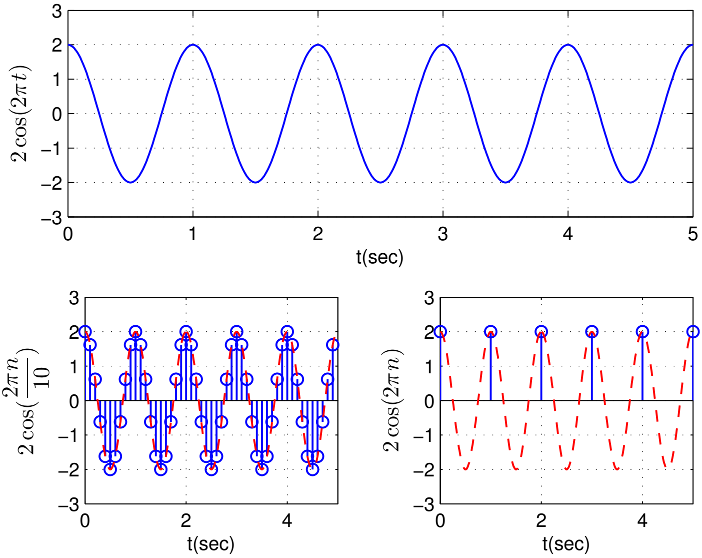
$$\{ \quad \cdots \quad x[-1] \quad x[0] \quad x[1] \quad x[2] \quad \cdots \}$$

according to the ordering of the samples. This process is called *sampling* or *discretization* of an analog signal.

Continuous-Time and Discrete-Time Representations

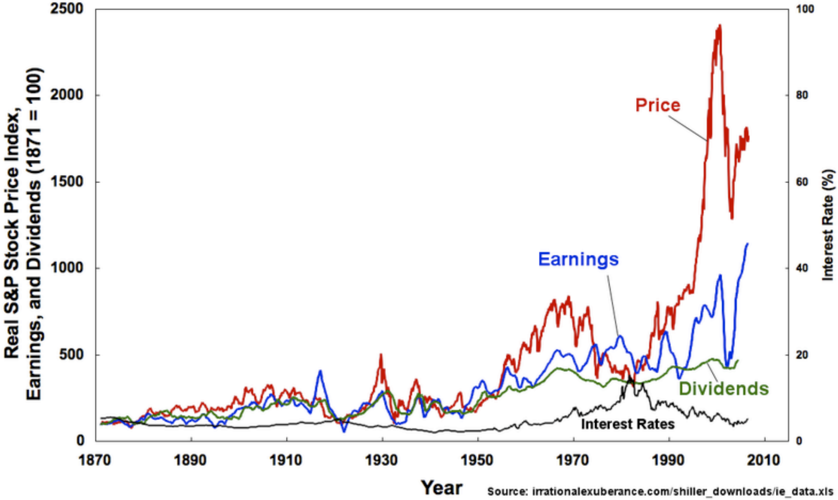
- By choosing a small value of T_s we could make the analog and the discrete-time signals look very similar but this is at the expense of memory space required to keep the numerous samples.
- If we make the value of T_s large, we improve the memory requirements, but at the risk of losing information contained in the original signal.

Continuous-Time and Discrete-Time Representations



Continuous-Time and Discrete-Time Representations

Natural discrete-time signal



Mathematics Background for Signals and Systems

- Derivatives and Finite Differences
- Integrals and Summations
- Differential and Difference Equations
- Complex Numbers
- Function of Complex Values

Derivatives and Finite Differences

- The derivative operator

$$D[x(t)] = \frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

  measure the rate of change of an analog signal $x(t)$.

- In finite calculus the *forward finite-difference operator*

$$\Delta[x(nT_s)] = x((n+1)T_s) - x(nT_s)$$

  measures the change in the signal from one sample to the next. If we let $x[n] = x(nT_s)$, for a known T_s, the forward finite-difference operator becomes a function of n:

$$\Delta[x[n]] = x[n+1] - x[n]$$

Derivatives and Finite Differences

- We have

$$\frac{dx(t)}{dt} = \lim_{T_s \to 0} \frac{\Delta[x(nT_s)]}{T_s}$$

- If the signal does not change very fast with respect to time, the finite-difference approximates well the derivative for relatively large values of T_s, but if the signal changes very fast one needs very small values of T_s.
- **example:** $x(t) = t^2$ sampling $x(t)$ with T_s, we have $x[n] = n^2$ and $\Delta[x[n]] = \Delta[n^2] = (n+1)^2 - n^2 = 2n + 1$ (Note: $x[n] = x(nT_s)$)

T_s	$dx/dt\|_{t=T_s n}$	$\Delta[x[n]]/T_s$		error	
		$n = 0$	$n = 1$	$n = 0$	$n = 1$
1	$2n$	1	3	1	1
0.01	$0.02n$	0.01	0.03	0.01	0.01

Integrals and Summations

The integration is the opposite of differentiation.

- $I(t) = \int_{t_0}^{t} x(\tau) d\tau$ The derivative of $I(t)$ is

$$\frac{dI(t)}{dt} = \lim_{h \to 0} \frac{I(t) - I(t-h)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} x(\tau) d\tau$$
$$\approx \lim_{h \to 0} \frac{x(t) + x(t-h)}{2} = x(t)$$

- thus, for a continuous signal $x(t)$,

$$\frac{d}{dt} \int_{t_0}^{t} x(\tau) d\tau = x(t)$$

- If using the derivative operator $D[\cdot]$, then its inverse $D^{-1}[\cdot]$ should be the integration operator. That is $D[D^{-1}[x(t)]] = x(t)$.

Integrals and Summations

Computationally, integration is implemented by sums.

- Consider, the integral of $x(t) = t$ from 0 to 10, which we know is equal to

$$\int_0^{10} t dt = \frac{t^2}{2}\Big|_{t=0}^{10} = 50.$$

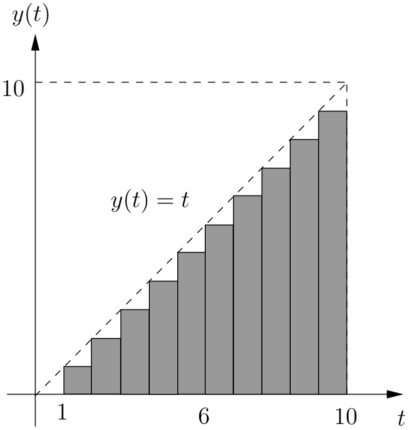
This area of a triangle with a base of 10 and a height of 10.

- Approximate the signal $x(t)$ by pulses $p[n]$ of width $T_s = 1$ and height $nT_s = n$, or pulses of area n for $n = 0, \ldots, 9$. The sum of the areas of the pulses is given by

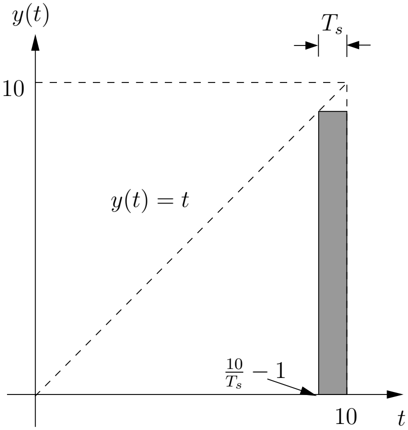
$$\sum_{n=0}^{9} p[n] = \sum_{n=0}^{9} nT_s = 0 + 1 + 2 + \cdots + 9 = 45$$

This approximation is very poor.

Integrals and Summations



(a) $T_s = 1$



(b) $T_s = 10^{-3}$

Integrals and Summations

- To improve the approximation of the integral we use $T_s = 10^{-3}$, which gives a discretized signal nT_s, for $0 \leq nT_s < 10$ or $0 \leq n \leq (10/T_s) - 1$. Then

$$\sum_{n=0}^{10^4-1} p[n] = \sum_{n=0}^{10^4-1} n10^{-6} = 49.995$$

  which is better result.

- In general for this case

$$\sum_{n=0}^{(10/T_s)-1} p[n] = \sum_{n=0}^{(10/T_s)-1} nT_s^2 = \frac{10(10-T_s)}{2}$$

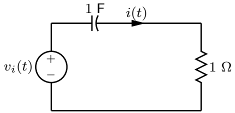
  for very small values of T_s (so that $10 - T_s \approx 10$) give $100/2 = 50$, as desired.

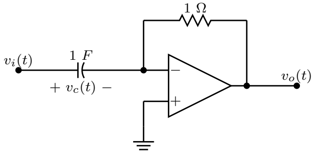
Differential and Difference Equations

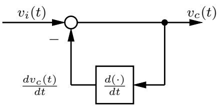
- A differential equation characterizes the dynamics of a continuous-time system, or the way the system responds to inputs over time.
- (Very) Long time ago we use a *analog computer* consists of operational amplifiers (op-amps), resisters, capacitors, voltage sources, and relays to solve these equations.
- Analog computer have gone the way of the dinosaurs, and it is digital computers aided by numerical methods used to solve differential equations.
- We need integrators to determine the solution of differential equation.

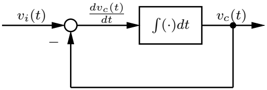
Differential and Difference Equations

RC circuit









The first order differential equation is given by

$$v_i(t) = v_c(t) + \frac{dv_c(t)}{dt}$$

with an initial voltage $v_c(0)$ across the capacitor. This equation is represented by the left-hand side block diagram.

- At the steady state ($t = \infty$), the voltage across the capacitor is equal to the voltage source–that is, the capacitor is acting as an open circuit given that the source is constant.
- If we want to know the transient state of $v_c(t)$, we need to solve the differential equation.
- Assuming that the source is switched on at time $t = 0$ and that the capacitor has an initial voltage $v_c(0)$, we have

$$v_c(t) = \int_0^t [v_i(\tau) - v_c(\tau)]\, d\tau + v_c(0), \qquad t \geq 0$$

- This is represented by the right-hand side of the previous block diagram.

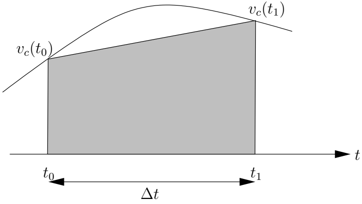
- By integrating

$$v_i(t) = v_c(t) + \frac{dv_c(t)}{dt},$$

we have

$$v_c(t_1) - v_c(t_0) = \int_{t_0}^{t_1} v_i(\tau)d\tau - \int_{t_0}^{t_1} v_c(\tau)d\tau$$

- If we let $t_1 - t_0 = \Delta t$ where $\Delta t \to 0$, the integrals can be seen as the area of small trapezoids of height Δt and bases $v_i(t_1)$ and $v_i(t_0)$ for the input source and $v_c(t_1)$ and $v_c(t_0)$ for the voltage across the capacitor.



- Using the formula for the area of a trapezoid we get an approximation for the integrals so that

$$v_c(t_1) - v_c(t_0) = [v_i(t_1) + v_i(t_0)]\frac{\Delta t}{2} - [v_c(t_1) + v_c(t_0)]\frac{\Delta t}{2}$$
$$v_c(t_1)\left[1 + \frac{\Delta t}{2}\right] = [v_i(t_1) + v_i(t_0)]\frac{\Delta t}{2} + v_c(t_0)\left[1 - \frac{\Delta t}{2}\right]$$

Assuming $\Delta t = T$, we let $t_1 = nT$ and $t_0 = (n-1)T$. The above equation can be written as

$$v_c(nT) = \frac{T}{2+T}\left[v_i(nT) + v_i((n-1)T)\right] + \frac{2-T}{2+T} v_c((n-1)T) \quad n \geq 1$$

and initial condition $v_c(0) = 0$. This is a first-order linear difference equation with constant coefficients approximating the differential equation characterizing the RC circuit. Letting the input be $v_i(t) = 1$ for $t \geq 0$, we have

$$v_c(nT) = \begin{cases} 0 & n = 0 \\ M + K v_c((n-1)T) & n \geq 1 \end{cases},$$

where $M = 2T/(2+T), K = (2-T)/(2+T)$. By using the recursive method, we obtain

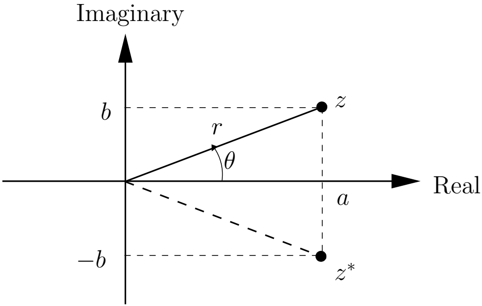
$$\begin{aligned} n = 0 \quad v_c(0) = 0 \qquad n = 1 \qquad v_c(T) = M \\ n = 2 \quad v_c(2T) = M + KM = M(1+K) \\ n = 3 \quad v_c(3T) = M + K(M + KM) = M(1 + K + K^2) \\ n = 4 \quad v_c(4T) = M + KM(1 + K + K^2) = M(1 + K + K^2 + K^3) \end{aligned}$$

- The response increases from the zero initial condition to a constant value, which is the effect of the dc source–the capacitor eventually acts as an open circuit, so that the voltage across the capacitor equals that of the input.
- Extrapolating from the above results, in the steady-state (i.e., when $nT \to \infty$) we have

$$v_c(nT) = M \sum_{m=0}^{\infty} K^m = \frac{M}{1-K} = 1$$

Complex Numbers

- A complex number z represents any point (a, b) in a two-dimensional plane by $z = a + jb$, where $a = \mathrm{Re}[z]$ is the **real part** and $b = \mathrm{Im}[z]$ is the **imaginary part**.



$$a = r\cos\theta \qquad b = r\sin\theta$$

$$z = a + jb = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta)$$

The Euler formula states that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

to prove the Euler formula, we expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$ using a Maclaurin series

$$\begin{aligned} e^{j\theta} &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \cdots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \end{aligned}$$

Hence $e^{j\theta} = \cos\theta + j\sin\theta$ and $z = a + jb = re^{j\theta}$.

Complex Numbers

A complex number can be expressed in Cartesian form $a + jb$ or polar form $re^{j\theta}$ with

$$a = r\cos\theta, \qquad b = r\sin\theta$$

and

$$r = \sqrt{a^2 + b^2}, \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

- r is the distance of the point z from the origin.
- r is called the **magnitude** of z and is denoted by $|z|$.
- θ is called the angle of z and is denoted by $\angle z$.
- Then we have $|z| = r, \qquad \angle z = \theta$ and $z = |z|e^{j\angle z}$

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z}$$

We define z^*, the **conjugate** of $z = a + jb$, as

$$\begin{aligned} z^* &= a - jb = re^{-j\theta} \\ &= |z|e^{-j\angle z} \end{aligned}$$

- the sum of a complex number and its conjugate is a real number equal to the real part of the number

$$z + z^* = (a + jb) + (a - jb) = 2a \text{ or } 2\,\mathrm{Re}\,z = z + z^*$$

- the subtraction of a complex number and its conjugate is a imaginary number with the magnitude equal to the imaginary part of the number

Complex Numbers

Conjugate of a Complex Number

- the product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number

$$zz^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2 \text{ or } |z| = \sqrt{zz^*}$$

- the division of a complex number z and its conjugate is

$$\frac{z}{z^*} = e^{j2\angle z} \text{ or } \angle z = -j0.5\,[\log(z) - \log(z^*)]$$

- for exaple

$$z = \frac{1 + j1}{3 + j4} = \frac{(1 + j1)(3 - j4)}{(3 + j4)(3 - j4)} = \frac{7 - j}{25}$$

Complex Numbers

the conversion of complex numbers

- the conversion of complex numbers from rectangular to polar needs to done with care.
- $z = 1 + j$ has a vector representing in the first quadrant of the complex plane, and its magnitude is $|z| = \sqrt{2}$ while the tangent of its angle θ is $\tan(\theta) = 1$ or $\theta = \pi/4$ radians.
- $z = -1 + j$, the vector representing it is now in the second quadrant with the same magnitude, but its angle is now $\theta = \pi - \tan^{-1}(1) = 3\pi/4$
- we find the angle with respect to the negative real axis and subtract it from π.
- if $z = -1 - j$, the magnitude does not change but the phase is now $\theta = \pi + \tan^{-1}(1) = 5\pi/4$ or $-3\pi/4$.

- One of the most famous equations of all times is

$$1 + e^{j\pi} = 1 + e^{-j\pi} = 0$$

  due to one of the most prolific mathematicians of all times, Leonard Euler (This guy is a Beethoven of Math world.)

- The Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- Consider the polar representation of the complex number $\cos\theta + j\sin\theta$
- it has a unit magnitude since

$$\sqrt{\cos^2\theta + \sin^2\theta} = 1 = \cos^2\theta + \sin^2\theta$$

- the angle of this complex number is $\psi = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] = \theta$ Thus, the complex number $\cos\theta + j\sin\theta = 1e^{j\theta}$, which is Euler's identity.
- When $\theta = \pm\pi$ the identity implies that $e^{\pm j\pi} = -1$, explaining the famous Euler's equation.

- Using Euler's identity the cosine and sine can be expressed as

$$\cos\theta = \mathrm{Re}\left[e^{j\theta}\right] = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \mathrm{Im}\left[e^{j\theta}\right] = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- We have

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Two complex number are given by $x = 5 + j74$ and $y = 2 - j4$. Calculate
(i) $\mathrm{Re}(x), \mathrm{Im}(x), \mathrm{Re}(y), \mathrm{Im}(y)$; (ii) $x + y$; (iii) $x - y$; (iv) xy; (v) x^*, y^*; (vi) $|x|, |y|$; and (vii) x/y.

Solution:

- $\mathrm{Re}(x) = 5$, $\mathrm{Im}(x) = 7$, $\mathrm{Re}(y) = 2$ and $\mathrm{Im}(y) = -4$.
- Adding x and y yields

$$x + y = (5 + j7) + (2 - j4) = (5 + 2) + j(7 - 4) = 7 + j3.$$

- Subtracting y from X

$$x - y = (5 + j7) - (2 - j4) = (5 - 2) + j(7 - (-4)) = 3 + j11.$$

- Multiplication of x and y is performed as follow:

$$\begin{aligned} xy &= (5 + j7)(2 - j4) = 10 + j14 - j20 - j^2 28 \\ &= (10 + 28) + j(14 - 20) = 38 - j6. \end{aligned}$$

  Multiplication is commutative, therefore $xy = yx$.

- The complex conjugate of the complex number $x = 5 + j7$ is $x^* = 5 - j7$. The complex conjugate of $y = 2 - j4$ is $y^* = 2 + j4$.
- $|x| = \sqrt{5^2 + 7^2} = \sqrt{74}$, $|y| = \sqrt{2^2 + (-4)^2} = \sqrt{20}$
- Dividing x by y yields

$$\begin{aligned} \frac{x}{y} &= \frac{5 + j7}{2 - j4} = \frac{5 + j7}{2 - j4} \cdot \frac{2 + j4}{2 + j4} \\ &= \frac{(5)(2) - (7)(4)}{2^2 + 4^2} + j\frac{(7)(2) + (5)(4)}{2^2 + 4^2} = -\frac{18}{20} + j\frac{34}{20}. \end{aligned}$$

- Consider a complex number $x = 2 + j4$. We have

$$r = \sqrt{2^2 + 4^2} = \sqrt{20} \text{ and } \theta = \tan^{-1}(4/2) = 0.35\pi \text{ radians.}$$

  The polar representation of $x = 2 + j4$ is $x = \sqrt{20}e^{j0.35\pi}$.
- Consider a complex number in the polar format $x = 4e^{j\pi/3}$. We have

$$a = r_x = 4\cos(\frac{\pi}{3}) = 2, \qquad b = r_y = 4\sin(\frac{\pi}{3}) = 2\sqrt{3} \Rightarrow x = 4e^{j\pi/3} = 2 + j2\sqrt{3}.$$

For $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$, find (i) $2z_1 - z_2$, (ii) $\frac{1}{z_1}$, (iii) $\frac{z_1}{z_2^2}$, (iv) $\sqrt[3]{z_2}$.

- We have to convert z_1 and z_2 to rectangular (Cartesian) form:

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = \sqrt{2} + j\sqrt{2}$$
$$z_2 = 8e^{j\pi/3} = 8\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 4 + j4\sqrt{3}$$

Therefore (i)

$$\begin{aligned} 2z_1 - z_2 &= 2(\sqrt{2} + j\sqrt{2}) - (4 + j4\sqrt{3}) = (2\sqrt{2} - 4) + j(2\sqrt{2} - 4\sqrt{3}) \\ &= -1.17 - j4.1 \end{aligned}$$

- (ii)

$$\frac{1}{z_1} = \frac{1}{2e^{j\pi/4}} = \frac{1}{2}e^{-j\pi/4}$$

- (iii)

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\frac{\pi}{4}-\frac{2\pi}{3})} = \frac{1}{32}e^{-j\frac{5\pi}{12}}$$

- (iv)

$$\sqrt[3]{z_2} = z_2^{1/3} = \left(8e^{j\pi/3}\right)^{1/3} = 2e^{j\pi/9}$$

Express the following numbers in polar form $(a) 2+j3, (b) -2+j1$.

- (a)

```
[rad,mag] = cart2pol(2,3)
rad = 0.9828
mag = 3.6056
deg = rad*(180/pi)
deg = 56.31
```

  therefore $z = 2 + j3 = 3.6056e^{j56.31^\circ}$.

- (b)

```
[angle,mag] = cart2pol(-2,1)
rad = 2.6779
mag = 2.2361
deg = rad*(180/pi)
deg = 153.4349
```

  therefore $z = -2 + j1 = 2.2361e^{j153.4349^\circ}$.

Complex Variable
MATLAB

Represent $4e^{-j\frac{3\pi}{4}}$ in rectangular form.

- ```
  [zreal,zimag] = pol2cart(-2*pi/4,4)
  zreal = -2.8284
  zimag = -2.8284
  ```
  Therefore

$$4e^{-j\frac{3\pi}{4}} = -2.8284 - j2.8284$$
  ```

Consider $F(\omega)$, a complex function of a real variable ω:

$$F(\omega) = \frac{2 + j\omega}{3 + j4\omega}$$

(a) Express $F(\omega)$ in rectangular form, and find its real and imaginary parts. (b) Express $F(\omega)$ in polar form, and find its magnitude $|F(\omega)|$ and angle $\angle F(\omega)$.

(a)

$$F(\omega) = \frac{(2 + j\omega)(3 - j4\omega)}{(3 + j4\omega)(3 - j4\omega)} = \frac{(6 + 4\omega^2) - j5\omega}{9 + 16\omega^2} = \frac{6 + 4\omega^2}{9 + 16\omega^2} - j\frac{5\omega}{9 + 16\omega^2}$$

Clearly the real and imaginary parts $F_r(\omega)$ and $F_i(\omega)$ are given by

$$F_r(\omega) = \frac{6 + 4\omega^2}{9 + 16\omega^2}, \qquad F_i(\omega) = \frac{-5\omega}{9 + 16\omega^2}$$

(b)

$$F(\omega) = \frac{2 + j\omega}{3 + j4\omega} = \frac{\sqrt{4 + \omega^2}e^{j\tan^{-1}\left(\frac{\omega}{2}\right)}}{\sqrt{9 + 16\omega^2}e^{j\tan^{-1}\left(\frac{4\omega}{3}\right)}} = \sqrt{\frac{4 + \omega^2}{9 + 16\omega^2}}e^{j\left[\tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)\right]}$$

Complex Variable
Examples

This is the polar representation of $F(\omega)$. Observe that

$$|F(\omega)| = \sqrt{\frac{4+\omega^2}{9+16\omega^2}}$$
$$\angle F(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)$$

Determine $z_1 z_2$ and $z1/z_2$ if $z_1 = 3 + j4$ and $z_2 = 2 + j3$ in rectangular form

```
z1 = 3+j*4; z2 = 2+j*3;
z1z2 = z1*z2
z1z2 = -6.000+17.0000i
z1_over_z2 = z1/z2
z1_over_z2 = 1.3486-0.0769i
```

Therefore

$$(3 + j4)(2 + j3) = -6 + j17 \text{ and } (3 + j4)/(2 + j3) = 1.3486 - 0.0769$$

Reference

1. Chaparro, L. F., *Signals and Systems using MATLAB*, Academic Press, 2011.
2. Lathi, B. P., *Signal Processing & Linear Systems*, Berkeley-Cambridge Press, 1998