INC 122 : RLC Circuits

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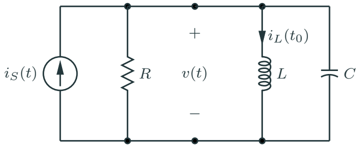
Learning Outcomes

Students should be able to:

- Determine the voltages and currents in the second-order transient circuits.
- Use Graphical and Symbolic tools to plot and check the calculation results.

Parallel RLC Circuit

Basic parallel RLC circuits



Using KCL, we have

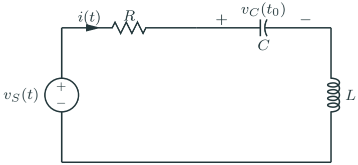
$$\frac{v(t)}{R} + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i_L(t_0) + C\frac{dv}{dt} = i_S(t)$$

Derivative with respect to t of both sides, we obtain

$$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{v(t)}{L} = \frac{di_S}{dt}$$

Series RLC Circuit

Basic series RLC circuits



Using KVL, we have

$$Ri(t) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v_C(t_0) + L\frac{di}{dt} = v_S(t)$$

Derivative with respect to t of both sides, we obtain

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i(t)}{C} = \frac{dv_S}{dt}$$

Both series and parallel circuits lead to a second-order differential equation with constant coefficients.

General Form of 2nd Order Circuit Equation

The RLC circuits, both parallel and series, have the same form equations:

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0x(t) = f(t)$$

If $x(t) = x_p(t)$, it is a solution of the general equation, and if $x(t) = x_c(t)$, it is a solution to the homogeneous equation

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_0x(t) = 0$$

Then

$$x(t) = x_p(t) + x_c(t)$$

is the solution of the general equation.

General Form of 2nd Order Circuit Equation

Using the fact that DC sources, i.e., $f(t) = A$ will reach all voltages and currents of the element is constant, i.e., $x_p(t) =$ constant. Then, by substituting this result back to the equation,

$$\begin{aligned}\frac{d^2 x_p(t)}{dt^2} + a_1 \frac{dx_p(t)}{dt} + a_2 x_p(t) &= A \\ a_0 x_p(t) &= A \\ x_p(t) &= \frac{A}{a_0} \\ x(t) &= \frac{A}{a_0} + x_c(t)\end{aligned}$$

The next step is to find the solution $x_c(t)$ to the homogeneous equation. Rewrite the equation in the form

$$\frac{d^2 x_c(t)}{dt^2} + 2\zeta\omega_n \frac{dx_c(t)}{dt} + \omega_n^2 x_c(t) = 0,$$

where $a_1 = 2\zeta\omega_n$ and $a_0 = \omega_n^2$.

General Form of 2nd Order Circuit Equation

To satisfy the homogenous equation, the frist and second-order derivatives of $x_c(t)$ must have the same form, hence

$$x_c(t) = Ke^{\lambda t}$$

and

$$\begin{aligned} \lambda^2 Ke^{\lambda t} + 2\zeta\omega_n \lambda Ke^{\lambda t} + \omega_n^2 Ke^{\lambda t} &= 0 \\ \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 &= 0 \text{ (note } Ke^{\lambda t} \neq 0 \ \forall t) \end{aligned}$$

We call

- $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ is called **characteristic equation.**
- ζ is called the **damping ratio.**
- ω_n is referred to as the **undamped natural frequency.**

General Form of 2nd Order Circuit Equation

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\begin{aligned}
\lambda &= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n - 4\omega_n^2}}{2} \\
&= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
\lambda_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad \lambda_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}
\end{aligned}$$

Both λ_i are the solution of the characteristic equation, then the complementary solution of the general from differential equation is of the form

$$x_c(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t},$$

and

$$x_c(0) = K_1 + K_2, \quad \left.\frac{dx_c}{dt}\right|_{t=0} = \lambda_1 K_1 + \lambda_2 K_2$$

$x_c(t)$ is an unforced response of the network (no input). There are three possible cases.

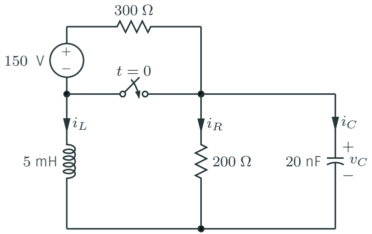
The Overdamped Circuit

Case 1 $\zeta > 1$ The term $\sqrt{\zeta^2 - 1}$ is greater than 0. The natural frequencies λ_1 and λ_2 are real and unequal. We have

$$x_c(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t}$$

This case is called **overdamped response**.

Example: Find an expression for $v_C(t)$ valid for $t > 0$.

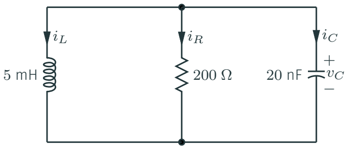


At $t < 0$:

$$v_C(0^-) = 150 \frac{200}{300 + 200} = 60 \text{ V}$$

$$i_L(0^-) = -\frac{150}{500} = -0.3 \text{ A}$$

The Overdamped Circuit



At $t > 0$, using KCL:

$$i_L + i_R + C\frac{dv_C}{dt} = 0$$
$$\frac{1}{L}\int_{t_0}^{t} v_C(\tau)d\tau + \frac{v_C}{R} + C\frac{dv_C}{dt} = 0$$
$$\frac{d^2v_C}{dt^2} + \frac{1}{RC}\frac{dv_C}{dt} + \frac{1}{LC}v_C(t) = 0$$

We have

$$2\zeta\omega_n = \frac{1}{RC} = 2.5 \times 10^5, \qquad \omega_n = \sqrt{\frac{1}{LC}} = 1 \times 10^5 \text{ rad/s}$$
$$2\zeta\omega_n = 2.5 \times 10^5, \quad \Rightarrow \quad \zeta = 1.25$$
$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = 1.25 \times 10^5 + 1 \times 10^5\sqrt{1.25^2 - 1}$$
$$= -5 \times 10^4, -2 \times 10^5$$

Then $v_C(t) = K_1 e^{-50000t} + K_2 e^{-200000t}$ V.

The Overdamped Circuit

We have

$$v_C(0^-) = 60 = K_1 + K_2$$

From

$$\begin{aligned}
i_L + i_R + C\frac{dv_C}{dt} &= 0 \\
\left.\frac{dv_C}{dt}\right|_{t=0^-} = \dot{v}_C(0^-) &= -\frac{i_L(0^-)}{C} - \frac{v_C(0^-)}{RC} \\
&= -\frac{-0.3}{20 \text{ nF}} - \frac{60}{200 \times 20 \text{ nF}} = 0 \\
&= -50000K_1 - 200000K_2
\end{aligned}$$

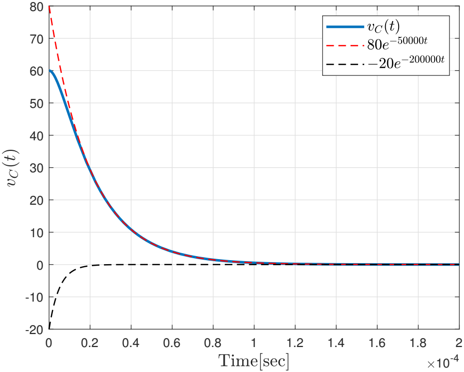
We have $K_1 = 80$ and $K_2 = -20$. So we obtain

$$v_C(t) = 80e^{-50000t} - 20e^{-200000t} \text{ V}$$

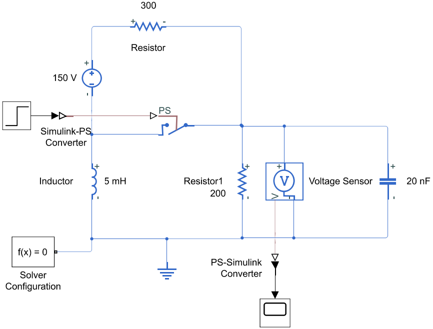
The Overdamped Circuit

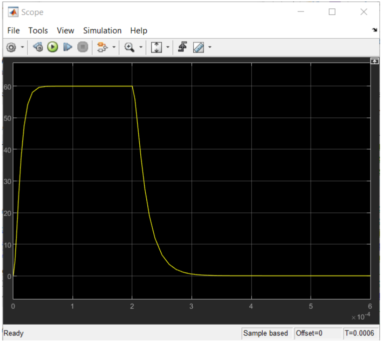
```
syms vc(t) t
R = 200; L = 5e-3; C = 20e-9;
eqn = diff(vc,t,2) + (1/(R*C))*diff(vc,t)
      + (1/(L*C))*vc(t) == 0;
Dvc = diff(vc,t);

cond1 = vc(0) == 60; cond2 = Dvc(0) == 0;
conds = [cond1; cond2];
vc = dsolve(eqn, conds);
```

```
yout = fplot(vc, [0, 2e-4],'linewidth', 2);
tt = yout.XData;
y1 = 80*exp(-5e4*tt);
y2 = -20*exp(-2e5*tt);
hold on
plot(tt, y1, 'r--', tt, y2, 'k--', 'linewidth', 1)
grid on ; hold off
```



The Overdamped Circuit





In Simscape the switch is closed at $t = 2 \times 10^{-4}$ sec.

The underdamped Circuit

Case 2 $\zeta < 1$ The term $\sqrt{\zeta^2 - 1}$ is less than 0. The roots of the characteristic equation can be written as

$$\begin{aligned}\lambda &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad \zeta^2 - 1 \text{ is negative} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega\end{aligned}$$

This case is called **underdamped response**. We have

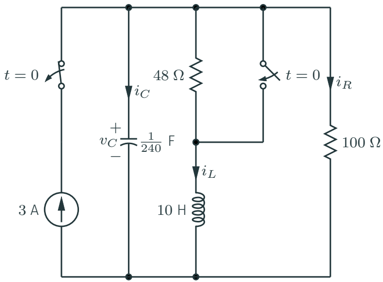
$$x_c(t) = Ce^{-\sigma t}\cos(\omega t + \theta), \quad \sigma = \zeta\omega_n \text{ and } \omega = \omega_n\sqrt{1-\zeta^2}$$

complex conjugate roots

$$\begin{aligned}x_c(t) &= K_1 e^{(-\sigma + j\omega)t} + K_2 e^{(-\sigma - j\omega)t} \\ K_1 &= \frac{C}{2}e^{j\theta}, \qquad K_2 = \frac{C}{2}e^{-j\theta} \\ x_c(t) &= \frac{C}{2}e^{-\sigma t}\left(e^{j(\omega t + \theta)} + e^{-(j\omega + \theta)}\right) \\ &= Ce^{-\sigma t}\cos(\omega t + \theta) \qquad \text{using Euler's identity}\end{aligned}$$

The underdamped Circuit

Determine $i_L(t)$ for the circuit.

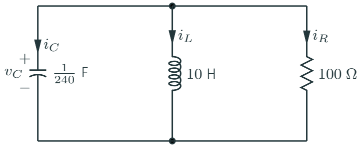


At $t < 0$, capacitor acts as an open circuit, and inductor acts as a short circuit.

$$i_L(0^-) = 3\left(\frac{100}{100+48}\right) = 2.027 \text{ A}$$

$$v_C(0^-) = 48 i_L(0^-) = 97.30 \text{ V}$$

At $t > 0$, the circuit is changed to $\Rightarrow$



The underdamped Circuit

Using KCL, we have

$$
\begin{aligned}
i_C + i_L + i_R &= 0 \\
C\frac{dv_C}{dt} + \frac{1}{L}\int_{-\infty}^{t} v_C(\tau)d\tau + \frac{v_C}{R} &= 0 \\
\frac{d^2 v_C}{dt^2} + \frac{1}{RC}\frac{dv_C}{dt} + \frac{1}{LC}v_C(t) &= 0
\end{aligned}
$$

Substuting all values, we have

$$
\frac{d^2 v_C}{dt^2} + 2.4\frac{dv_C}{dt} + 24 v_C(t) = 0,
$$

where

$$
\begin{aligned}
\omega_n = \sqrt{24} = 4.899 \text{ rad/sec}, \quad 2\zeta\omega_n = 2.4 \;\Rightarrow\; \zeta = 0.245 \\
\lambda = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -1.2 \pm j4.75
\end{aligned}
$$

The underdamped Circuit

$$v_C(t) = Ke^{-1.2t}\cos(4.75t + \theta)$$
$$v_C(0^-) = 97.30 = K\cos(\theta)$$

Since $i_C + i_L + i_R = 0$, we have

$$\left.\frac{dv_C}{dt}\right|_{t=0-} = -\frac{i_L(0^-)}{C} - \frac{v_C(0^-)}{RC} = -486.48 - 233.5200 = -720$$

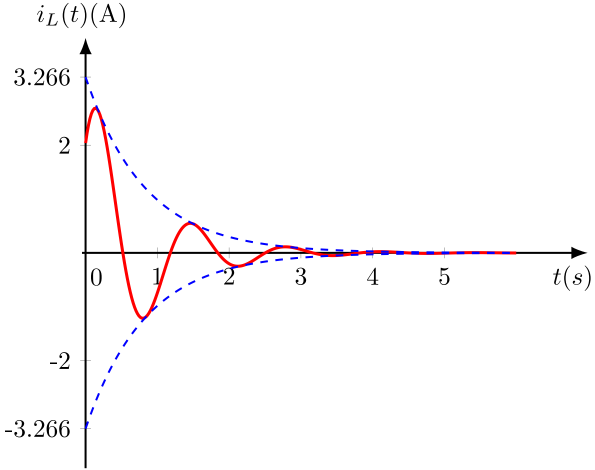
Then,

$$\left.\frac{dv_C}{dt}\right|_{t=0-} = \dot{v}_C(0^-) = -720 = -1.2K\cos(\theta) - 4.75K\sin(\theta)$$
$$K\sin(\theta) = (-720 + 1.2(97.3))/(-4.75) = 127$$
$$K = \sqrt{97.3^2 + 127^2} = 160$$
$$\theta = \tan^{-1}\frac{127}{97.30} = 52.54^\circ$$
$$v_C(t) = 160e^{-1.2t}\cos(4.75t + 52.54^\circ)$$

Note that,

$$
\begin{aligned}
i_L(t) &= -\frac{1}{240}\frac{dv_C}{dt} - \frac{v_C}{100} \\
&= -\frac{1}{240}\left(-192e^{-1.2t}\cos(4.75t + 52.54^\circ) - 760e^{-1.2t}\sin(4.75t + 52.54)\right) \\
&\quad - 1.6e^{-1.2t}\cos(4.75t + 52.54^\circ) \\
&= -0.8e^{-1.2t}\cos(4.75t + 52.54^\circ) + 3.167e^{-1.2t}\sin(4.75t + 52.54^\circ) \\
&= -0.8e^{-1.2t}\left(\cos(4.75t)\cos(52.54^\circ) - \sin(4.75t)\sin(52.54^\circ)\right) \\
&\quad + 3.167e^{-1.2t}\left(\sin(4.75t)\cos(52.54^\circ) + \sin(52.54^\circ)\cos(4.75t)\right) \\
&= e^{-1.2t}\left(-0.4866\cos(4.75t) + 0.6350\sin(4.75t)\right) \\
&\quad + e^{-1.2t}\left(1.9262\sin(4.7t) + 2.5139\cos(4.75t)\right) \\
&= e^{-1.2t}\left(2.0273\cos(4.75t) + 2.5612\sin(4.75t)\right) \\
&= 3.266e^{-1.2t}\cos(4.75t - 51.34^\circ)
\end{aligned}
$$

The underdamped Circuit



The critically damped Circuit

Case 3 $\zeta = 1$ The term $\sqrt{\zeta^2 - 1}$ is 0. The roots of the characteristic equation are repeated as

$$\lambda_1 = \lambda_2 = -\zeta\omega_n$$

This case is called **critically damped** response. We have

$$x_c(t) = B_1 e^{\lambda_1 t} + B_2 t e^{\lambda_2 t}.$$

Proof*

Consider the following case

$$\frac{d^2 x(t)}{dt^2} + 2\alpha \frac{dx(t)}{dt} + \alpha^2 x(t) = 0$$

Since the roots are equal to α, the homogenous solution could be (wrong!)

$$x_c(t) = K_1 e^{-\alpha t} + K_2 e^{-\alpha t} = K_3 e^{-\alpha t}$$

Proof*

One know solution is

$$x_{c1}(t) = K_3 e^{-\alpha t} \qquad \text{and} \qquad x_{c2}(t) = x_{c1}(t)v(t) = K_3 e^{-\alpha t} v(t)$$

The equation becomes

$$\frac{d^2}{dt^2}[K_3 e^{-\alpha t} v(t)] + 2\alpha \frac{d}{dt}[K_3 e^{-\alpha t} v(t)] + \alpha^2 K_3 e^{-\alpha t} v(t) = 0$$

Evaluating the derivatives, we obtain

$$\frac{d}{dt}[K_3 e^{-\alpha t} v(t)] = -K_3 \alpha e^{-\alpha t} v(t) + K_3 e^{-\alpha t} \frac{dv(t)}{dt}$$
$$\frac{d^2}{dt}[K_3 e^{-\alpha t} v(t)] = K_3 \alpha^2 e^{-\alpha t} v(t) - 2K_3 \alpha e^{-\alpha t} \frac{dv(t)}{dt} + K_3 e^{-\alpha t} \frac{d^2 v(t)}{dt}$$

The critically damped Circuit

Proof*

Substituting these expressions into the preceding equation yields

$$K_3 e^{-\alpha t} \frac{d^2 v(t)}{dt^2} = 0$$

Therefore,

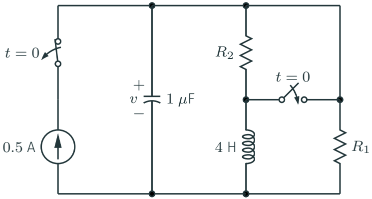
$$\frac{d^2 v(t)}{dt^2} = 0 \;\Rightarrow\; v(t) = A_1 + A_2 t$$

Therefore, the general solution is

$$x_{c2}(t) = x_{c1}(t)v(t) = K_3 e^{-\alpha t}[A_1 + A_2 t] = B_1 e^{\lambda t} + B_2 t e^{\lambda t}$$

The critically damped Circuit

Select a value for R_1 such that the circuit in Fig. below will be characterized by a critically damped response for $t > 0$, and a value for R_2 such that $v(0^-) = 100$ V. Fine $v(t)$ at $t = 1$ ms.

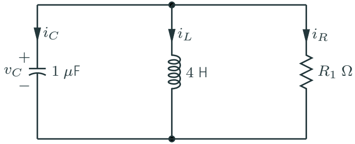


At $t < 0$, capacitor acts as an open circuit, and inductor acts as a short circuit.

$$i_L(0^-) = 0.5\left(\frac{R_1}{R_1 + R_2}\right) \quad \text{A}$$

$$v(0^-) = 0.5\left(\frac{R_1 R_2}{R_1 + R_2}\right) \quad \text{V}$$

At $t > 0$, the circuit is changed to ⇒



The critically damped Circuit

Using KCL, we have

$$i_C + i_L + i_R = 0$$
$$\frac{d^2 v_C}{dt^2} + \frac{1}{R_1 C}\frac{dv_C}{dt} + \frac{1}{LC} v_c(t) = 0$$

To have a critically damped response, we have to have $(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0, \zeta = 1)$

$$\frac{1}{2R_1C} = \sqrt{\frac{1}{LC}} = 500$$
$$R_1 = \frac{1}{10 \times 10^3(1 \times 10^{-6})} = 1000\ \Omega$$

We need $v(0^-) = 100$ V, so

$$v(0^-) = 0.5\left(\frac{R_1R_2}{R_1 + R_2}\right) = 100 \quad \Longrightarrow \quad R_2 = 250\ \Omega$$
$$i_L(0^-) = 0.4 \text{ A}$$

The critically damped Circuit

Note we have $\lambda = \frac{-1}{2R_1C} = -500$, then

$$v(t) = B_1 e^{-500t} + B_2 t e^{-500t}$$
$$v(0^-) = 100 = B_1$$

From $i_C + i_L + i_R = 0$, we have

$$\left.\frac{dv}{dt}\right|_{t=0^-} = \dot{v}_C(0^-) = \frac{i_L(0^-)}{C} - \frac{v_C(0^-)}{R_1C} = -5 \times 10^5 \text{ V/s}$$

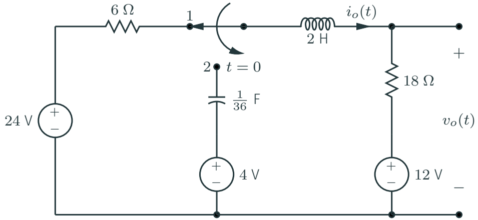
From
$$\frac{dv}{dt} = -5 \times 10^4 e^{-500t} + B_2 e^{-500t} - 500 B_2 t e^{-500t}$$
$$\left.\frac{dv}{dt}\right|_{t=0^-} = \dot{v}_C(0^-) = -5 \times 10^5 = -5 \times 10^4 + B_2$$
$$B_2 = -4.5 \times 10^5$$

Then

$$v(t) = 100e^{-500t} - 4.5 \times 10^5 t e^{-500t} \text{ and } v(t = 1 \times 10^{-3}) = -212.2856 \text{ V.}$$

RLC Circuit: Hard Example

The switch in the network in Fig. below moves from position 1 to position 2 at $t = 0$. Compute $i_0(t)$ for $t > 0$ and use this current to determine $v_o(t)$ for $t > 0$.



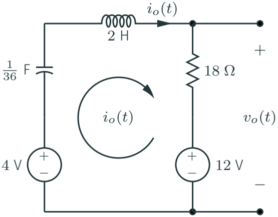
At $t < 0$, we have

$$i_L(0^-) = i_o(0^-) = \frac{24 - 12}{24} = 0.5 \text{ A}$$

$$v_C(0^-) = 0 \text{ V}$$

$$v_o(0^-) = 0.5(18) = 9 + 12 = 21 \text{ V}$$

At $t > 0$



Using KVL, we have

$$-4 + 36\int_{-\infty}^{t} i_o(\tau)d\tau + 2\frac{di_o}{dt} + 18i_o(t) + 12 = 0$$

$$\frac{d^2 i_o}{dt^2} + 9\frac{di_o}{dt} + 18i_o(t) = 0$$

RLC Circuit: Hard Example

At the steady-state, the capacitor acts as an open circuit. Then $i_{op}(t)$ (particular solution) is zero, and $i_{op}(t) = 0$. Considering the transient response, we obtain

$$\frac{d^2 i_o}{dt^2} + 9\frac{di_o}{dt} + 18 i_o(t) = 0$$

$$\lambda_{1,2} = -\frac{9}{2} \pm \frac{\sqrt{81-72}}{2} = -\frac{9}{2} \pm \frac{3}{2} = -3, -6$$

We have a complementary solution

$$i_{oc}(t) = K_1 e^{-3t} + K_2 e^{-6t} \quad \Rightarrow \quad i_{oc}(0^-) = 0.5 = K_1 + K_2$$

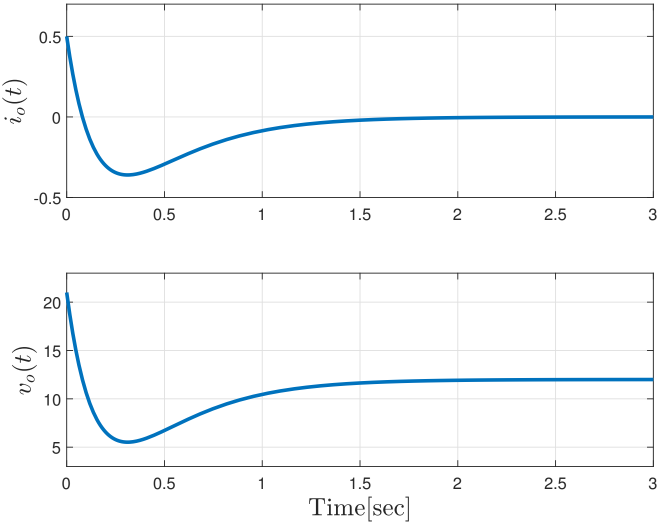
Since $\left.\frac{di_o}{dt}\right|_{t=0^-} = -9 i_o(0^-) - \frac{v_C(0^-)}{2} - 4 = -\frac{17}{2}$, then

$$-\frac{17}{2} = -3K_1 - 6K_2 \quad \Rightarrow \quad K_1 = -\frac{11}{6}, \quad K_2 = \frac{14}{6}$$

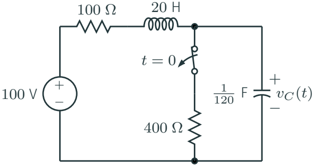
$$i_{oc}(t) = i_o(t) = -\frac{11}{6} e^{-3t} + \frac{14}{6} e^{-6t} \text{ A}$$

$$v_o(t) = 12 + 18 i_o(t) = 12 - 33 e^{-3t} + 42 e^{-6t} \text{ V}$$

RLC Circuit: Hard Example (Matlab Plot)



RLC Circuit: Hard Example II

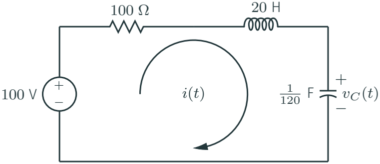


Find $v_C(t)$ at time $t > 0$.

At $t = 0^-$, we have

$$i_L(0^-) = \frac{100}{100 + 400} = 0.2 \text{ A}, \quad v_C(0^-) = \frac{100(400)}{100 + 400} = 80 \text{ V}$$

The circuit at time $t > 0$.



$$Ri(t) + L\frac{di}{dt} + v_C(t) = 100, \qquad i_C(t) = C\frac{dv_C}{dt}$$

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C(t) = \frac{100}{LC}$$

$$\frac{d^2 v_C}{dt^2} + 5\frac{dv_C}{dt} + 6v_C(t) = 600$$

RLC Circuit: Hard Example II

The particular solution is

$$6v_{Cp}(t) = 600 \quad \Longrightarrow \quad v_{Cp}(t) = 100 \text{ V}$$

The characteristic solution is

$$\lambda^2 + 5\lambda + 6 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = -2, -3$$

Then the complementary solution is

$$v_{Cc}(t) = K_1 e^{-2t} + K_2 e^{-3t} \quad \text{and} \quad v_C(t) = 100 + K_1 e^{-2t} + K_2 e^{-3t} \text{ V}$$

We have

$$v_C(0^-) = 80 \text{ V} \quad \text{and} \quad i_C(0^-) = C\frac{dv_C}{dt}\bigg|_{0^-} \Longrightarrow \dot{v}_C(0^-) = \frac{0.2}{1/120} = 24 \text{ V}$$

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Thus

$$100 + K_1 + K_2 = 80 \quad \text{and} \quad -2K_1 - 3K_2 = 24$$
$$K_1 = -36, \quad K_2 = 16$$

We obtain

$$v_C(t) = 100 - 36e^{-2t} + 16e^{-3t} \text{ V}$$

Reference

1. William H. Hayt, Jr., Jack E. Kemmerly, and Steven M. Durbin *Engineering Circuit Analysis*, 8th Edition McGraw-Hill, 2012.
2. J. David Irwin, and R. Mark Nelms *Basic Engineering Circuit Analysis*, 11th, Wiley, 2015.