INC 122 : RL and RC Circuits

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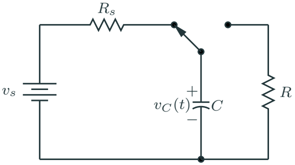
Learning Outcomes

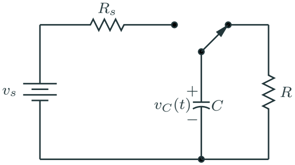
Students should be able to:

- Calculate the initial values for inductor currents and capacitor voltages in the transient circuits.
- Determine the voltages and currents in the first-order transient circuits.
- Use Graphical and Symbolic tools to plot and check the calculation results.l

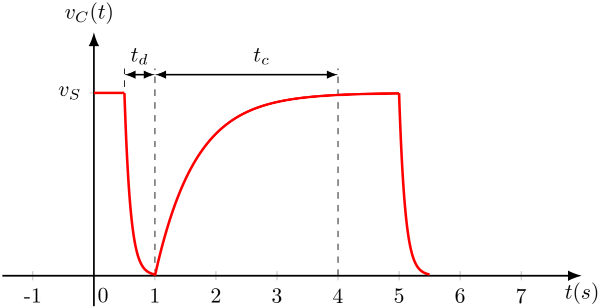
RC Circuits application: Camera's Flash Circuit



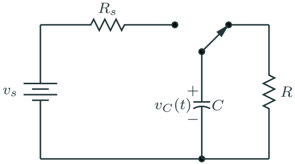
Charge period
t_c



Discharge period
t_d



RC Circuits application: Discharge



Discharge period
t_d

KCL for the circuit

$$C\frac{dv_C}{dt} + \frac{v_C(t)}{R} = 0$$

$$\frac{dv_C}{dt} + \frac{1}{RC}v_C(t) = 0$$

$$v_C(t) = V_0 e^{-\frac{1}{RC}t}$$

- The solution function is a decaying exponential.
- The rate at which it decays is a function of the values of R and C.
- The product RC is a very important parameter, called **time constant** τ.

First-Order Circuit: General Form

A first-order differential equation:

$$\frac{dx}{dt} + ax(t) = f(t)$$

There are two solutions for this problem:

- $x(t) = x_p(t)$ is any solution to the general equation. $x_p(t)$ is called the **particular integral solution**, or **forced response**.
- $x(t) = x_c(t)$ is any solution to the homogeneous equation

$$\frac{dx}{dt} + ax(t) = 0.$$

  $x_c(t)$ is called the **complementary solution**, or **natural response**.

If we consider the situation in which $f(t) = A$ (some constant). The general solution $x(t)$ consists of two parts that are obtained by solving the two equations

$$\frac{dx_p}{dt} + ax_p(t) = A$$
$$\frac{dx_c}{dt} + ax_c(t) = 0$$

First-Order Circuit: General Form

Since

$$\frac{dx_p}{dt} + ax_p(t) = A,$$

It is reasonable to assume that the solution $x_p(t)$ must also be a constant. We have

$$x_p(t) = K_1 \Rightarrow K_1 = \frac{A}{a}$$

From

$$\frac{dx_c}{dt} + ax_c(t) = 0$$

We have

$$\frac{1}{x_c(t)} dx_c = -a \quad \Rightarrow \quad \ln x_c(t) = -at + C$$
$$x_c(t) = K_2 e^{-at}$$

Thus $x(t) = x_p(t) + x_c(t) = \frac{A}{a} + K_2 e^{-at}$. In general case, $x(t) = K_1 + K_2 e^{-\frac{1}{\tau}t}$.

First-Order Circuit: General Form

Consider the general solution

$$x(t) = K_1 + K_2 e^{-\frac{1}{\tau}t}$$

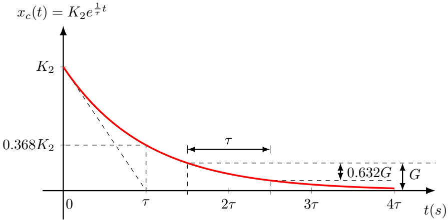
Each part of the equation has a names that are commonly employed in electrical engineering.

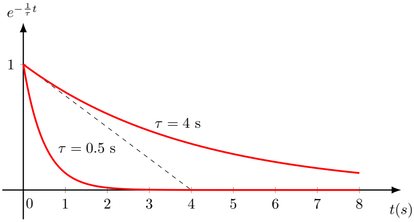
- Term K_1 is referred to as the **steady-state solution**: the value of the variable $x(t)$ as $t \to \infty$, the second term become zero.
- The constant τ is called the **time constant** of the circuit. The second term is a decaying exponential.

$$K_2 e^{-\frac{1}{\tau}t} = \begin{cases} K_2, & \tau > 0 \text{ and } t = 0 \\ 0, & \tau > 0 \text{ and } t = \infty \end{cases}$$

- The rate at which the exponential decays is determined by the time constant τ.

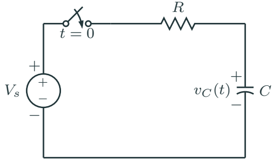
First-Order Circuit: General Form





- The value of $x_c(t)$ has fallen from K_2 to a value of $0.368K_2$ in one time constant, a drop of 63.2%.
- In two time constants the value of $x_c(t)$ has fallen to $0.135K_2$, a drop of 63.2% from the value at time $t = \tau$, and the final value of the curve is closed by 63.2% each time constant.
- After five time constants, $x_c(t) = 0.0067K_2$, which is less than 1%
- The circuit with a small-time constant has a fast response, and a large time constant circuit has a slow response.

Analysis Techniques RC Circuit: Differential Equations



Using KCL for $t > 0$ is

$$C\frac{dv_C}{dt} + \frac{v_C(t) - V_s}{R} = 0$$

$$\frac{dv_C}{dt} + \frac{1}{RC}v_C(t) = \frac{V_s}{RC}$$

From the previous section, we have

$$v_C(t) = K_1 + K_2 e^{-\frac{1}{\tau}t}$$

Substituting the solution into the differential equation yields

$$-\frac{K_2}{\tau}e^{-\frac{1}{\tau}t} + \frac{K_1}{RC} + \frac{K_2}{RC}e^{-\frac{1}{\tau}t} = \frac{V_s}{RC}$$

Equating the constant and exponential terms, we obtain

$$K_1 = V_s \qquad \text{and} \qquad \tau = RC$$

Analysis Techniques RC Circuit: Differential Equations

Therefore

$$v_C(t) = V_s + K_2 e^{-\frac{1}{RC}t}$$

To find the value of K_2, we need to know the initial condition of $v_C(0^-)$. Here the capacitor is uncharged at $t < 0$, then

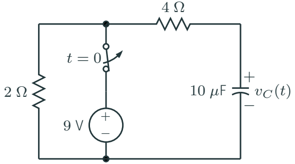
$$0 = V_s + K_2 \qquad \Rightarrow \qquad K_2 = -V_s$$

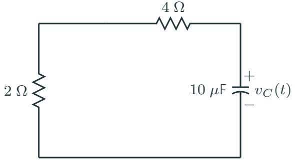
Hence, the complete solution for the voltage $v_C(t)$ is

$$v_C(t) = V_s(1 - e^{\frac{1}{RC}t}).$$

Since $\tau = RC$ we can change the time constant by changing the value of RC.

Analysis Techniques RC Circuit: Example





At $t < 0$, we have $v_C(0^-) = 9$ V.

We have

$$C\frac{dv_C}{dt} + \frac{v_C(t)}{R} = 0$$

$$v_C(t) = v_C(0^-)e^{-\frac{1}{RC}t}$$

$$= 9e^{-\frac{1}{60\times 10^{-6}}t} \text{ V}$$

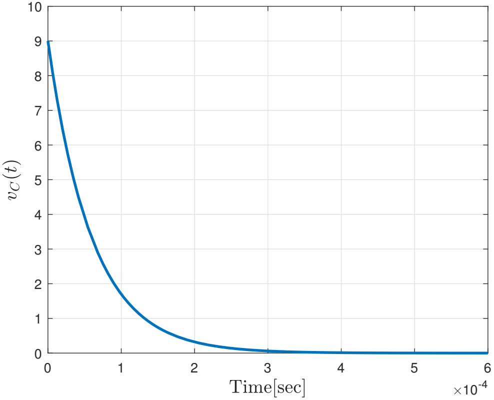
Note: The differential equation of this question is

$$\frac{dv_C}{dt} + \frac{1}{(6)(10 \times 10^{-6})}v_C(t) = 0$$

We can solve this problem by using Symbolic computational program.

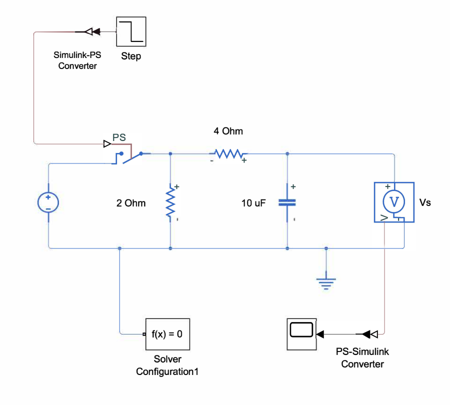
Analysis Techniques RC Circuit: Example (Matlab)

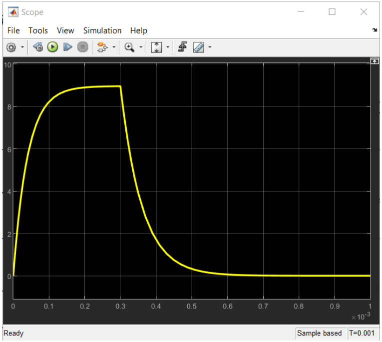
```
syms vc(t) t
R = 6; C = 10e-6;
eqn = diff(vc,t) + (1/(R*C))*vc(t) == 0;

vc = dsolve(eqn, vc(0)== 9);
fplot(vc, [0, 0.6e-3])
```



$$v_C(t) = 9e^{-\frac{50000}{3}t}$$

Analysis Techniques RC Circuit: Example (Simscape)

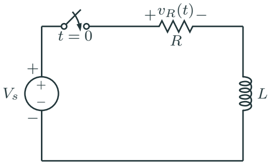




Consider $v_C(t)$ after the fully charge period from $t > 0.3$ ms.

Analysis Techniques RL Circuit: Differential Equation

Determine $v_R(t)$ of the circuit below at time $t > 0$.



Using KVL for $t > 0$, we have

$$L\frac{di_L}{dt} + Ri_L(t) = V_s$$
$$\frac{di_L}{dt} + \frac{R}{L}i_L(t) = \frac{V_s}{L}$$

From the standard from, we have

$$i_L(t) = K_1 + K_2 e^{-\frac{1}{\tau}t}$$

Substituting the solution into the differential equation yields

$$-\frac{K_2}{\tau}e^{-\frac{1}{\tau}t} + \frac{R}{L}K_1 + \frac{R}{L}K_2 e^{-\frac{1}{\tau}t} = \frac{V_s}{L}$$

Equating the constant and exponential terms, we obtain

$$K_1 = \frac{V_s}{R} \qquad \text{and} \qquad \tau = \frac{L}{R}$$

Analysis Techniques RL Circuit: Differential Equation

Therefore

$$i_L(t) = \frac{V_s}{R} + K_2 e^{-\frac{R}{L}t}.$$

To find the value of K_2, we need to know the initial condition of $i_L(0^-)$. Since there are no initial current in the inductor at $t < 0$, then

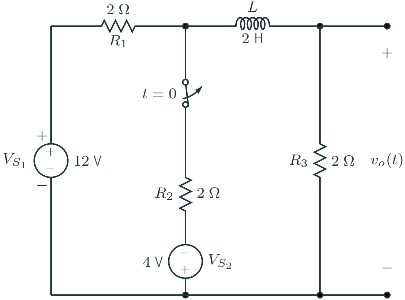
$$0 = \frac{V_s}{R} + K_2 \qquad \Rightarrow \qquad K_2 = -\frac{V_s}{R}$$

Hence, the complete solution for the current $i_L(t)$ is

$$i_L(t) = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$
$$v_R(t) = Ri_L(t) = V_s(1 - e^{-\frac{R}{L}t}).$$

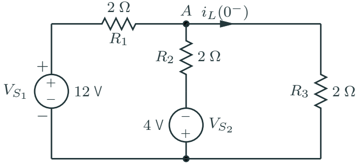
Since $\tau = \frac{L}{R}$ we can change the time constant by changing the value of R or L.

Analysis Techniques RL Circuit: Example



The switch in the network opens at $t = 0$. Let us find the output voltage $v_o(t)$ for $t > 0$.

The circuit at $t < 0$



There are several ways to find $i_L(0^-)$. Here we use KCL.

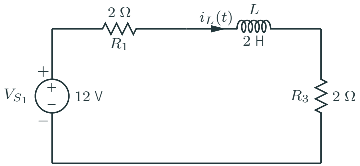
$$\frac{v_A - 12}{2} + \frac{v_A + 4}{2} + \frac{v_A}{2} = 0$$

$$\frac{3}{2}v_A = 4 \;\Rightarrow\; v_A = \frac{8}{3}\text{ V}$$

$$i_L(0^-) = \frac{4}{3}\text{ A}$$

Analysis Techniques RL Circuit: Example

The circuit at $t > 0$



$$\frac{di_L}{dt} + 2i_L(t) = 6$$

$$i_L(t) = K_1 + K_2 e^{-\frac{R}{L}t}$$

$$= K_1 + K_2 e^{-2t}$$

$$K_1 = 3 \text{ and } \tau = 0.5 \text{ s}$$

Since $i_L(0^-) = \frac{4}{3}$ A, we have

$$\frac{4}{3} = 3 + K_2 \Rightarrow K_2 = -\frac{5}{3}$$

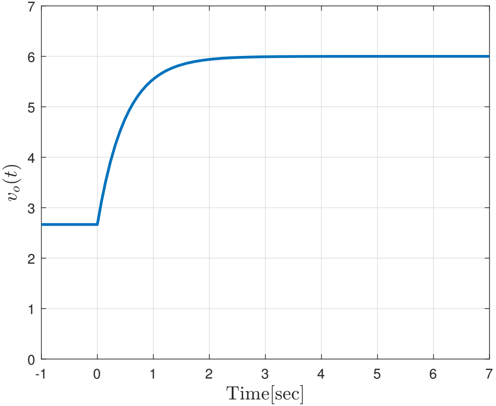
Thus,

$$i_L(t) = 3 - \frac{5}{3}e^{-2t} \text{ A} \Rightarrow v_o(t) = 6 - \frac{10}{3}e^{-2t} \text{ V}$$

Analysis Techniques RL Circuit: Example (Matlab)

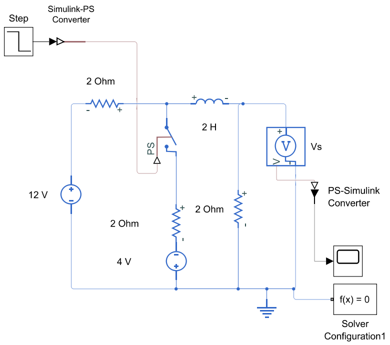
```
syms iL(t) t
R = 4; L = 2;
eqn = diff(iL,t) + (R/L)*iL(t) == 12/L;

iL = dsolve(eqn, iL(0)== 4/3);
vo = 2*iL
yout = fplot(vo, [0, 7])
```

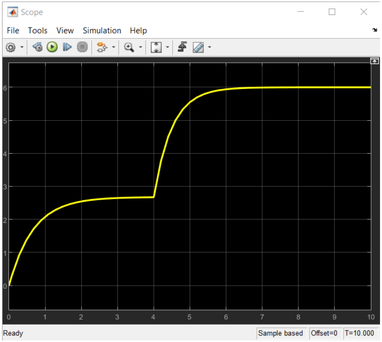
```
dt = yout.XData(2)- yout.XData(1);
tx = -1:dt:0;
tt = [tx yout.XData];
yy = [yout.YData(1)*ones(size(tx)) yout.
    YData];
plot(tt,yy,'linewidth', 2)
```



$$v_o(t) = 6 - \frac{10}{3}e^{-2t}$$

Analysis Techniques RL Circuit: Example (Simscape)





Consider $v_o(t)$ from $t > 3$ s.

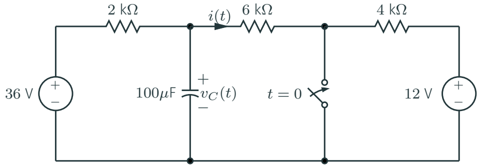
Analysis Techniques RC and RL Circuits

- We will not consider the step-by-step method. It is no benefit.
- To use the step-by-step method, we need to store more formulas, which are not necessary.
- Simple using KVL and KCL analysis are more than enough.
- Just keep in your mind that

$$v_C(0^-) = v_C(0^+) = v_C(0)$$
$$i_L(0^-) = i_L(0^+) = i_L(0)$$

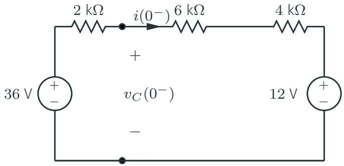
This phenomenon is from the physical behavior of inductors and capacitors.

Analysis Techniques RC Circuit: (Hard) Example



Consider the circuit shown in Fig. The circuit is in steady state prior to time $t = 0$, when the switch is closed. Let us calculate the current $i(t)$ for $t > 0$.

Firstly, we need to find $i(0^-)$ and $v_C(0^-)$ as follow:



Find $i(0^-)$:

$$i(0^-) = \frac{36 - 12}{12 \times 10^3} = 2 \text{ mA}$$

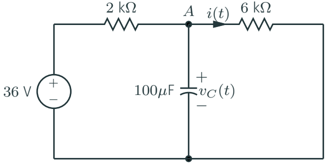
Find $v_C(0^-)$

$$-36 + 2 \times 10^3(2 \times 10^{-3}) + v_C(0^-) = 0$$

$$v_C(0^-) = 32 \text{ V}$$

Analysis Techniques RC Circuit: (Hard) Example

The circuit at $t > 0$:



Using KCL

$$\frac{v_C - 36}{2 \times 10^3} + C\frac{dv_C}{dt} + \frac{v_C}{6 \times 10^3} = 0$$

$$100 \times 10^{-6}\frac{dv_C}{dt} + \frac{4v_C(t)}{6 \times 10^3} = \frac{108}{6 \times 10^3}$$

$$\frac{dv_C}{dt} + \frac{20}{3}v_C(t) = 180$$

From the standard from we have

$$v_C(t) = K_1 + K_2 e^{-\frac{20}{3}t}\ \text{V},\ K_1 = 27, \qquad \tau = \frac{3}{20}\ \text{s}$$

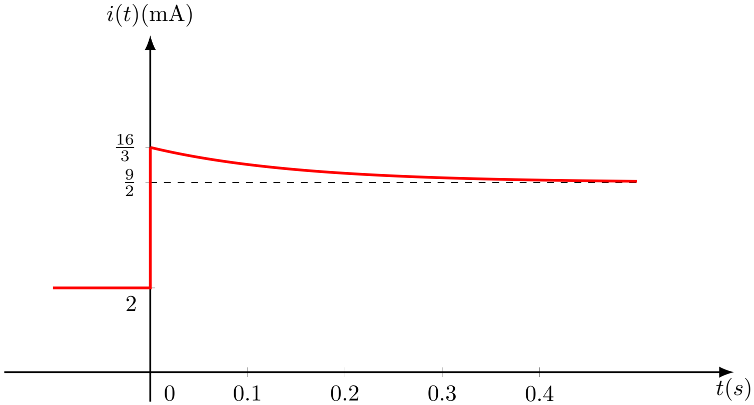
$$v_C(t) = 27 + K_2 e^{-\frac{20}{3}t}$$

$$32 = 27 + K_2 \quad \Rightarrow \quad K_2 = 5$$

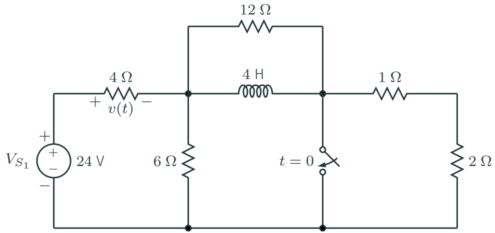
Thus

$$i(t) = \frac{1}{6 \times 10^3}v_C(t) = \frac{9}{2} + \frac{5}{6}e^{-\frac{20}{3}t}\ \text{mA}$$

Analysis Techniques RC Circuit: (Hard) Example

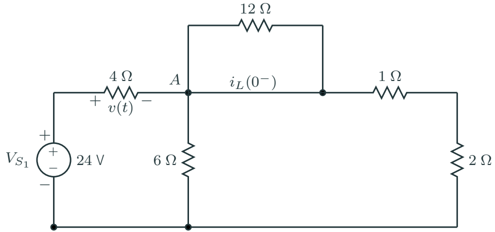


Analysis Techniques RC Circuit: (Hard) Example II



The circuit shown in Fig is assumed to have been in a steady-state condition prior to switch closure at $t = 0$. We wish to calculate the voltage $v(t)$ for $t > 0$.

We need to start to find $i_L(0^-)$ and $v(0^-)$.



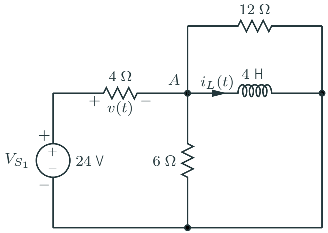
$$v_A(0^-) = \frac{24(2)}{2+4} = 8 \text{ V}$$

$$i_L(0^-) = 4\left(\frac{6}{6+3}\right) = \frac{8}{3} \text{ A}$$

$$v(0^-) = \frac{24(4)}{2+4} = 16 \text{ V}$$

Analysis Techniques RC Circuit: (Hard) Example II

The circuit at $t > 0$:



Find $i_L(t)$ as follow:

$$\frac{v_A - 24}{4} + \frac{v_A}{6} + i_L(t) + \frac{v_A}{12} = 0$$

$$v_A + 2i_L(t) = 12$$

$$\frac{di_L}{dt} + \frac{1}{2}i_L(t) = 3, \quad v_A = L\frac{di_L}{dt}$$

From the standard from we have

$$i_L(t) = K_1 + K_2 e^{-\frac{1}{2}t}, \quad K_1 = 6, \quad \tau = 2\ s$$

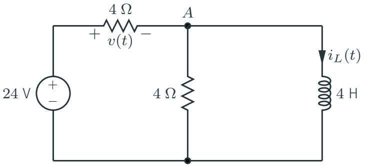
$$i_L(t) = 6 + K_2 e^{-\frac{1}{2}t}$$

$$\frac{8}{3} = 6 + K_2 \ \Rightarrow \ K_2 = -\frac{10}{3}$$

$$i_L(t) = 6 - \frac{10}{3}e^{-\frac{1}{2}t}\ \text{A}$$

Analysis Techniques RC Circuit: (Hard) Example II

Find $v(t)$



By using KVL:

$$-24 + v(t) + v_A = 0$$

$$v(t) = 24 - v_A = 24 - 4\frac{di_L}{dt}$$

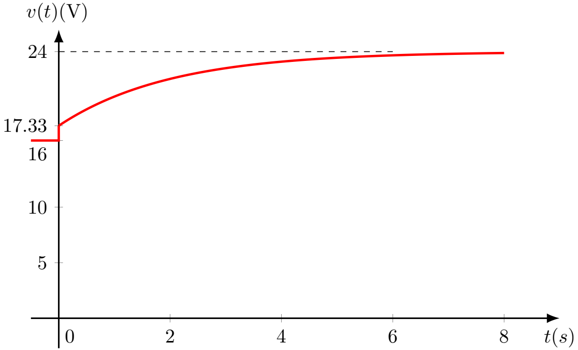
$$v(t) = 24 - \frac{20}{3}e^{-\frac{1}{2}t} \text{ V}$$

Since

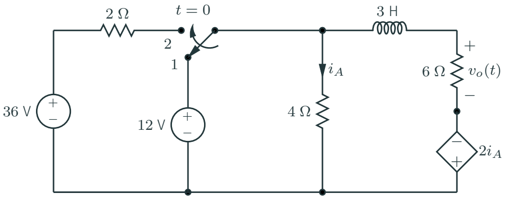
$$\frac{di}{dt} = \frac{d}{dt}\left(6 - \frac{10}{3}e^{-\frac{1}{2}t}\right) = \frac{5}{3}e^{-\frac{1}{2}t}$$

$$v_A(t) = v_L(t) = L\frac{di_L}{dt}$$

Analysis Techniques RC Circuit: (Hard) Example II

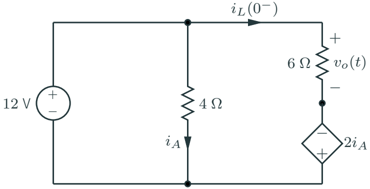


Analysis Techniques RC Circuit: (Hard) Example III



The circuit shown in Fig has reached steady state with the switch in position 1. At time $t = 0$ the switch moves from position 1 to position 2. We want to calculate $v_o(t)$ for $t > 0$.

We need to start to find $i_L(0^-)$ and $v_o(0^-)$.



$$i_A = \frac{12}{4} = 3 \text{ A}$$

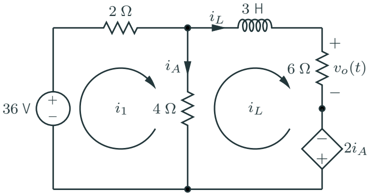
$$v_o(0^-) - 2(3) = 12$$

$$v_o(0^-) = 18 \text{ V}$$

$$i_L(0^-) = \frac{12 + 2(3)}{6} = 3 \text{ A}$$

Analysis Techniques RC Circuit: (Hard) Example III

The circuit at $t > 0$:



Find $i_L(t)$ as follow:

$$-36 + 2i_1 + 4(i_1 - i_L) = 0$$

$$i_1 = 6 + \frac{2}{3}i_L$$

$$i_A = 6 - \frac{1}{3}i_L$$

$$4(i_L - i_1) + L\frac{di_L}{dt} + 6i_L - 2i_A = 0$$

$$\frac{di_L}{dt} + \frac{8}{3}i_L = 12$$

From the standard from we have

$$i_L(t) = K_1 + K_2 e^{-\frac{8}{3}t}, \quad K_1 = \frac{9}{2}, \quad \tau = \frac{3}{8}\ s$$

$$i_L(t) = \frac{9}{2} + K_2 e^{-\frac{8}{3}t}$$

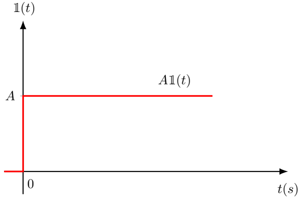
$$3 = \frac{9}{2} + K_2 \ \Rightarrow\ K_2 = -\frac{3}{2}$$

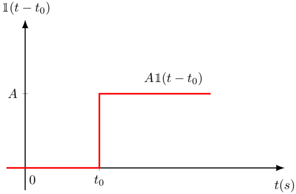
$$i_L(t) = \frac{9}{2} - \frac{3}{2}e^{-\frac{8}{3}t}\ \text{A} \ \Rightarrow\ v_o(t) = 6i_L(t) = 27 - 9e^{-\frac{8}{3}t}\ \text{V}$$

Pulse Response

- When a voltage or current source in a circuit is suddenly applied, the voltages or currents in the circuit are forced to change abruptly.
- A forcing function whose value changes in a discontinuous manner or has a discontinuous derivative is called a **singular function**.
- There are two important singular functions in circuit analysis: the unit step impulse function and the unit step function.
- The **unit step function** is defined as

$$\mathbb{1}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}, \qquad \mathbb{1}(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t \geq t_0 \end{cases}$$

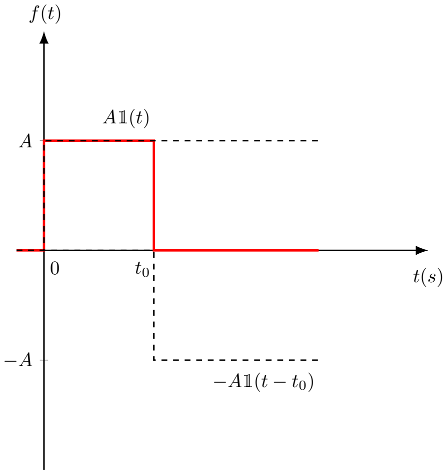




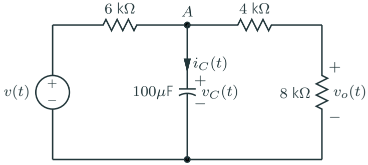
Pulse Response

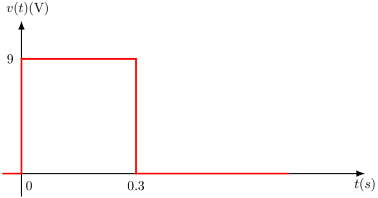
- Pulse function

$$f(t) = A\left[\mathbb{1}(t) - \mathbb{1}(t - t_0)\right]$$



Analysis Techniques RC Circuit: Pulse Example





Determine the expression for the voltage $v_o(t)$.

Note: $v_C(0^-) = v_o(0^-) = 0$ V

Use KCL at point A, we have

$$\frac{v_C - 9}{6\ k\Omega} + i_C(t) + \frac{v_C}{12\ k\Omega} = 0$$

$$i_C(t) = C\frac{dv_C}{dt}$$

$$\frac{dv_C}{dt} + \frac{5}{2}v_C(t) = 15$$

$$\frac{dv_C}{dt} + \frac{5}{2}v_C(t) = 15 \quad \Rightarrow \quad v_C(t) = K_1 + K_2 e^{-\frac{1}{\tau}t}$$

$$K_1 = \frac{30}{5} = 6, \qquad \tau = \frac{2}{5}$$

$$v_C(t) = 6 + K_2 e^{-\frac{5}{2}t} \quad \Rightarrow \quad v_C(0^-) = 0 \;\Rightarrow\; K_2 = -6$$

Thus,

$$v_C(t) = 6\left(1 - e^{-\frac{5}{2}t}\right)$$

By using voltage divider technique, we have

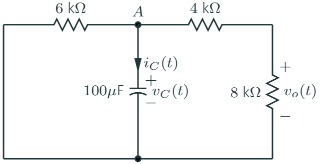
$$v_o(t) = \frac{2}{3}v_C(t) = 4\left(1 - e^{-\frac{5}{2}t}\right)$$

At $t = 0.3$, we obtain

$$v_C(0.3) = 6\left(1 - e^{-\frac{5}{2}(0.3)}\right) = 3.16 \quad \text{and} \quad v_o(0.3) = 3.16\frac{2}{3} = 2.11 \text{ V}$$

Analysis Techniques RC Circuit: Pulse Example

After $t > 0.3$,



$$v_C(0.3) = 3.16 \text{ V}$$
$$v_o(0.3) = 2.11 \text{ V}$$

$$\frac{v_C(t)}{6 \text{ k}\Omega} + i_C(t) + \frac{v_C(t)}{12 \text{ k}\Omega} = 0$$

$$\frac{dv_C}{dt} + \frac{5}{2}v_C(t) = 0, \qquad v_C(t) = K_1 + K_2 e^{-\frac{5}{2}(t-0.3)}$$

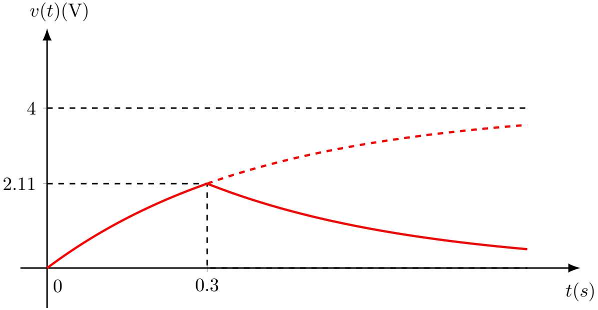
$$K_1 = 0, \qquad K_2 = 3.16$$

$$v_C(t-0.3) = 3.16e^{-\frac{5}{2}(t-0.3)}$$

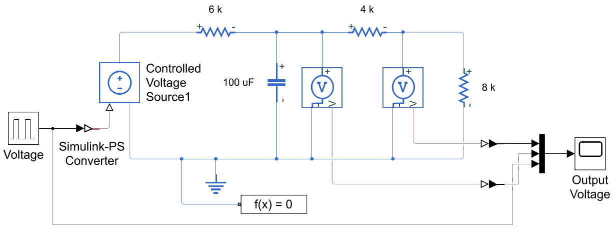
$$v_o(t-0.3) = \frac{2}{3}v_C(t-0.3) = 2.11e^{-\frac{5}{2}(t-0.3)} \ V$$

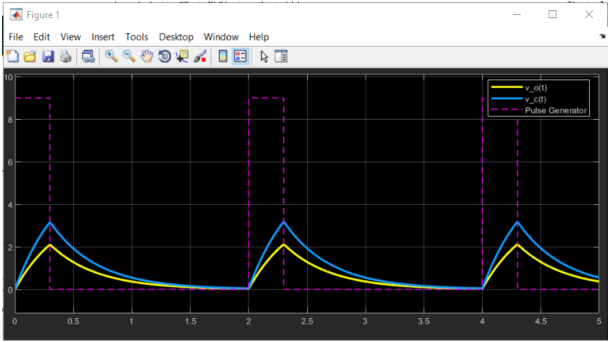
$$v_o(t) = \begin{cases} 0, & t \leq 0 \\ 4\left(1 - e^{-\frac{5}{2}t}\right), & 0 \leq t < 0.3 \\ 2.11e^{-\frac{5}{2}(t-0.3)}, & t \geq 0.3 \end{cases}$$

Analysis Techniques RC Circuit: Pulse Example



Analysis Techniques RC Circuit: Pulse Example





Reference

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2. J. David Irwin, and R. Mark Nelms *Basic Engineering Circuit Analysis*, 11th, Wiley, 2015.