INC 122 : Capacitor and Inductor

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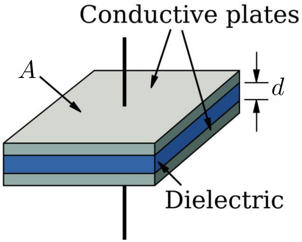
Introduction

This section introduces two new passive circuit elements:

- Capacitor and Inductor.
- They can both store and deliver finite amounts of energy.
- They are not ideal sources since they cannot sustain a finite average power flow over an infinite time interval.
- They are classed as linear elements.
- The current-voltage relationships for them are time-dependent.
- They use in a wide range of modern circuit applications.

Capacitor



The capacitance of two parallel plates of area A, separated by distance d is

$$C = \frac{\varepsilon_0 A}{d},$$

where ε_0, the permitivity of free space, is 8.85×10^{-12} F/m.

If $d = 1.016 \times 10^{-4}$ m the thickness of none sheet of oil-impregnated paper, to make a 100 F capacitance, then

$$100 \text{ F} = \frac{8.85 \times 10^{-12} A}{1.016 \times 10^{-4}}$$
$$A = 1.148 \times 10^{9} \text{m}^2,$$

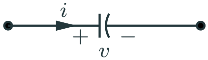
which is $A \approx 7.175 \times 10^5$ rai

Capacitor: Ideal Capacitor Model

- The capacitor is a passive circuit element. We define capacitance C by the voltage-current relationship defined by

$$i(t) = C\frac{dv}{dt},$$

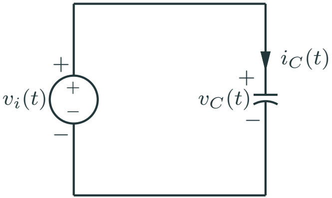
where v and i are voltage and current.



Electrical symbol

- Capacitance is measured in coulombs per volt or farads. The unit **farad (F)** is name after Michael Faraday.
- Capacitors may be fixed or variable and typically range from thousands of microfarads (μF) to a few picofarads (pF).

Capacitor: Ideal Capacitor Model

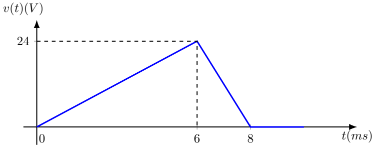


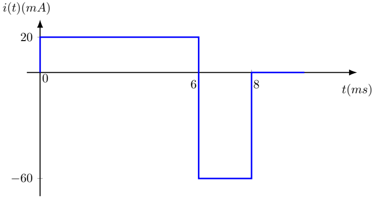
Determine the current $i_C(t)$ flowing through the capacitor if $C = 2$ F.

- $v_i(t) = 5\ V$
- $v_i(t) = \cos(5t)\ V$
- $v_i(t) = 2e^{-5t}\ V$

Capacitor: Ideal Capacitor Model

The voltage across a $5\ \mu$F capacitor has the waveform below. Determine the current wave form.





Note that

$$v(t) = \begin{cases} 0\text{ V} & , t < 0\ \text{ ms} \\ \frac{24}{6\times10^{-3}} t\text{ V} & , 0 \leq t \leq 6\text{ ms} \\ \frac{-24}{2\times10^{-3}} t + 96\text{ V} & , 6 \leq t < 8\text{ ms} \\ 0\text{ V} & , 8\text{ ms} \leq t \end{cases}$$

$$i(t) = C\frac{dv(t)}{dt} = \begin{cases} 20\text{ mA} & , 0 \leq t \leq 6\text{ ms} \\ -60\text{ mA} & , 6 \leq t < 8\text{ ms} \\ 0\text{ mA} & , 8\text{ ms} \geq t \end{cases}$$

Capacitor: Integral Voltage-Current Relationships

- The capacitor voltage may be expressed in terms of the current by

$$i(t) = C\frac{dv}{dt} \Rightarrow dv = \frac{1}{C}i(t)dt,$$

then we have

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0) \Rightarrow v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

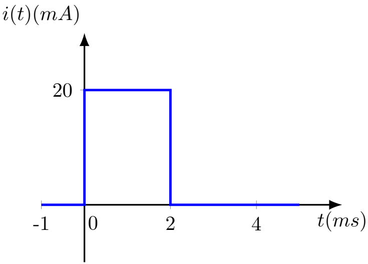
- Since the integral of the current over any time interval is the corresponding charge accumulated on the capacitor plate into which the current is flowing, we can define capacitance as

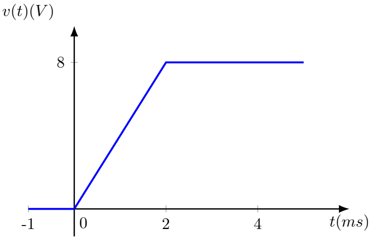
$$q(t) = Cv(t),$$

where $q(t)$ and $v(t)$ represent instantaneous values of the charge on either plate and the voltage between the plates.

Capacitor: Integral Voltage-Current Relationships

Find the capacitor voltage that is associated with the current shown graphically in the plot below. The value of the capacitor is $5\ \mu$F.





Using

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0), \quad v(t_0) = v(-\infty) = v(0) = 0$$

$$v(t) = \frac{1}{5\times 10^{-6}}\int_{0}^{t} 20\times 10^{-3} d\tau + 0$$

$$= \begin{cases} 4000t \text{ V}, & 0 \le t \le 2 \text{ ms} \\ 8 \text{ V}, & t \ge 2 \text{ ms} \end{cases}$$

Capacitor: Energy Storage

The energy stored in the capacitor can be derived from the power that is delivered to the element. The power is given by

$$p(t) = v(t)i(t) = Cv(t)\frac{dv}{dt}$$

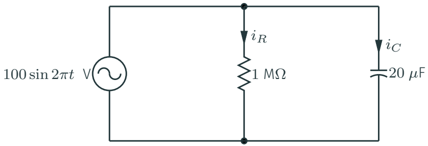
The energy stored in the electric field is

$$\begin{aligned} w_C(t) &= \int_{t_0}^{t} p(\tau)d\tau = C\int_{t_0}^{t} v(\tau)\frac{d(v(\tau))}{d\tau}d\tau \\ &= C\int_{v(t_0)}^{v(t)} v(\tau)dv(\tau) = \frac{1}{2}C\left[v^2(t) - v^2(t_0)\right] \\ &= \frac{1}{2}Cv^2(t) \text{ Joul,} \qquad \text{if } v(t_0) = 0 \text{ V} \end{aligned}$$

Since $q = Cv$, we also have

$$w_C(t) = \frac{1}{2}\frac{q^2(t)}{C} \text{ Joul}$$

Capacitor: Energy Storage



Find the maximum energy stored in the capacitor and the energy dissipated in the resistor over the interval $0 < t < 0.5$ s.

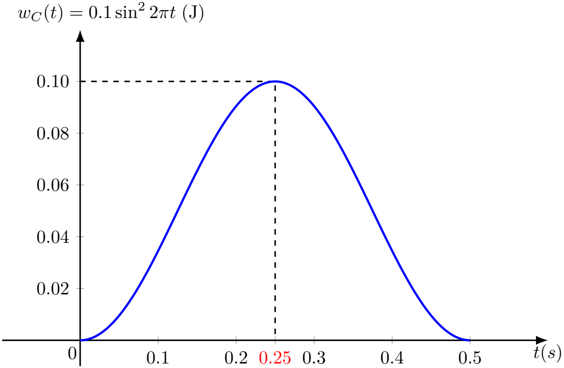
The energy stored in the capacitor is

$$w_C(t) = \frac{1}{2} C v^2(t) = 0.1 \sin^2 2\pi t \quad J$$

and $i_R = v(t)/R = 10^{-4} \sin 2\pi t$ A and $p_R = i_R^2 R = 10^{-2} \sin^2 2\pi t$ W, so the energy dissipated in the resistor is

$$\begin{aligned} w_R &= \int_0^{0.5} p_R dt = \int_0^{0.5} 10^{-2} \sin^2 2\pi t dt \\ &= \frac{10^{-2}}{2} \int_0^{0.5} (1 - \cos 4\pi t)\, dt = \frac{10^{-2}}{2} (0.5 - 0) = 2.5 \text{ mJ} \end{aligned}$$

Capacitor: Energy Storage

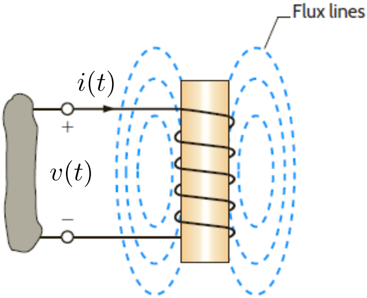


The energy stored in the capacitor increases from zero at $t = 0$ to a maximum of 100 mJ at $t = 0.25$ s and then decreases to zero in another $t = 0.25$ s. Thus, $w_{C_{\text{max}}} = 100$ mJ.

Capacitor: Important Characteristics of an Ideal Capacitor

- There is no current through a capacitor if the voltage across it is not changing with time. A capacitor is therefore and *open circuit to DC*.
- A finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero, such as when the voltage across it is constant.
- It is impossible to change the voltage across a capacitor by a finite amount is zero time, as this requires an infinite current through the capacitor. (A capacitor resists an abrupt change in the voltage across it in a manner analogous to the way a spring sesists and abrupt change in its displacement.)
- A capacitor never dissipates energy, but only stores it. Although this is true for the *mathematical model*, it is not true for a *physical* capacitor due to finite resistances associated with the dielectric as well as the packaging.

Inductor



The electric current flowing through a conductor generates a magnetic field surrounding it. The magnetic flux linkage φ generated by a given current $i(t)$ depends on the geometric shape of the circuit. Their ratio defines the **inductance** L, Thus

$$L = \frac{d\varphi}{di}$$

- An inductor or "coil" that has the form of a long helix of very small pitch is found to have and inductance of $\mu N^2 A/s$ where A is the cross-sectional area, s is the axial length of the helix, N is the number of complete turns of wire, and μ is a constant of the material inside the helix, called permeability.

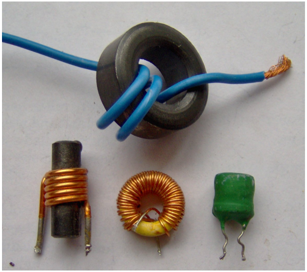
Inductor: Ideal Inductor Model

- The inductor is a passive circuit element. We define inductance L by the voltage-current relationship defined by

$$v(t) = L\frac{di}{dt},$$

where v and i are voltage and current.

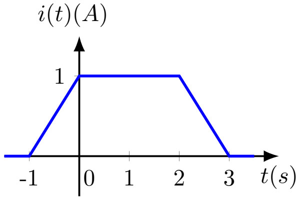
Electrical symbol

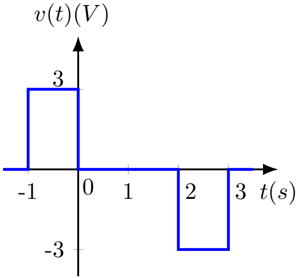


- Inductance is measured in volt-second per ampere. The unit **henry (H)** is name after 19th century American scientist Joseph Henry.

Inductor: Ideal Inductor Model

Given the waveform of the current in a 3 H inductor as shown in the plot, determine the inductor voltage and sketch it.





Note that

$$i(t) = \begin{cases} t+1 \text{ A} & , -1 \leq t \leq 0 \text{ s} \\ 1 \text{ A} & , 0 \leq t < 2 \text{ s} \\ -t+3 \text{ A} & , 2 \leq t < 3 \text{ s} \end{cases}$$

$$v(t) = L\frac{di(t)}{dt} = \begin{cases} 3 \text{ A} & , -1 \leq t \leq 0 \text{ s} \\ 0 \text{ A} & , 0 \leq t < 2 \text{ s} \\ -3 \text{ A} & , 2 \leq t < 3 \text{ s} \end{cases}$$

Inductor: Integral Voltage-Current Relationships

- The inductor current can be expressed in terms of the voltage by

$$v(t) = L\frac{di}{dt} \Rightarrow di = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0) \Rightarrow i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

- The voltage across a 2 H inductor is known to be $6\cos 5t$ V. Determine the resulting inductor current if $i(t = -\pi/2) = 1$ A.

$$\begin{aligned} i(t) &= \frac{1}{2}\int_{t_0}^{t} 6\cos 5\tau d\tau + i(t_0) \\ &= \frac{1}{2}\left(\frac{6}{5}\right)\sin 5t - \frac{1}{2}\left(\frac{6}{5}\right)\sin 5t_0 + i(t_0) \\ &= 0.6\sin 5t \underbrace{-0.6\sin 5t_0 + i(t_0)}_{\text{constant}} = 0.6\sin 5t + k \end{aligned}$$

We have

$$i\left(\frac{-\pi}{2}\right) = 1 = 0.6\sin\left(-5\frac{\pi}{2}\right) + k$$

$$k = 1 + 0.6 = 1.6,$$

then

$$i(t) = 0.6 \sin 5t + 1.6 \quad \text{A}$$

- Note: we cannot use

$$i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau) d\tau$$

because $0.6 \sin(\pm\infty)$ is indeterminate. We don't know what is $i(t_0)$.

Inductor: Energy Storage

The energy stored in the inductor can be derived from the power that is delivered to the element. The power is given by

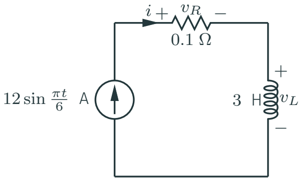
$$p(t) = v(t)i(t) = Li(t)\frac{di}{dt}$$

The energy stored in the magnetic field is

$$\begin{aligned} w_L(t) &= \int_{t_0}^{t} p(\tau)d\tau = L\int_{t_0}^{t} i(\tau)\frac{d(i(\tau))}{d\tau}d\tau \\ &= L\int_{i(t_0)}^{i(t)} i(\tau)di(\tau) = \frac{1}{2}L[i^2(t) - i^2(t_0)] \\ &= \frac{1}{2}Li^2(t) \ \text{ Joul }, \quad \text{if } i(t_0) = 0 \end{aligned}$$

Inductor: Energy Storage

Find the maximum energy stored in the inductor, and calculate how much energy is dissipated in the resistor in the time during which the energy is being stored in, and then recovered from, the inductor.

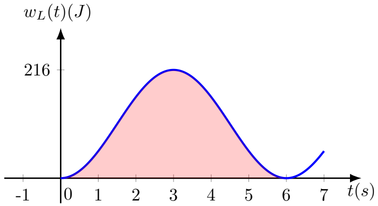


The energy stored in the inductor is

$$w_L = \frac{1}{2}Li^2 = 216\sin^2\frac{\pi t}{6}\ \text{J}$$

The peak of the sine wave is at $t = 3$ sec, to get $216\sin^2\frac{\pi}{2}$. Thus the maximum energy stored in the inductor is 216 J.

Inductor: Energy Storage



From $w_L(t)$, we can see that we store and remove the energy in 6 seconds. The power dissipated in the resistor is easily found as

$$p_R = i^2 R = 14.4 \sin^2 \frac{\pi t}{6}$$

and the energy converted into heat in the resistor within this 6 second interval is

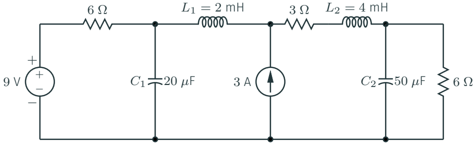
$$\begin{aligned} w_R &= \int_0^6 p_R dt = \int_0^6 14.4 \sin^2 \frac{\pi}{6} t dt \\ &= \int_0^6 14.4 \left(\frac{1}{2}\right) \left(1 - \cos \frac{\pi}{3} t\right) dt = 43.2 \text{ J} \end{aligned}$$

The 43.2 J is 20% of the maximum stored energy (216 J), which is big because of the large inductance. For coil having an inductance of about 100 μH, we might expect a figure closer to 2 or 3 percent.

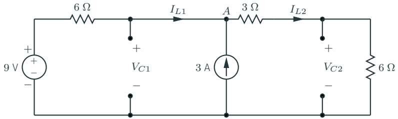
Inductor: Important Characteristics of an Ideal Inductor

- There is no voltage across an inductor if the current through it is not changing with time. An inductor is therefore a *short circuit to DC*.
- A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero, such as when the current through it is constant.
- It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor. (An inductor resists an abrupt change in the current through it in a manner analogous to the way a mass resists an abrupt change in its velocity.)
- The inductor never dissipates energy, but only stores it. Although this is true for the *mathematical* model, it is not true for a *physical* inductor due to series resistances.

Find the total energy stored in the circuit.



The circuit has only DC sources. Then, we replace the capacitors with open circuits and the inductors with short circuits.



Using KCL at node A we get

$$I_{L2} = I_{L1} + 3$$

Energy Storage

Using KVL around the outside of the circuit yields

$$6I_{L1} + 3I_{L2} + 6I_{L2} = 9$$

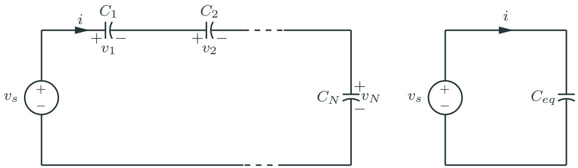
We have $I_{L1} = -1.2$ A and $I_{L2} = 1.8$ A. The voltages V_{C1} and V_{C2} can be calculated from

$$V_{C1} - 9 + 6I_{L1} = 0 \Rightarrow V_{C1} = 16.2 \text{ V}$$
$$-V_{C2} + 6I_{L2} = 0 \Rightarrow V_{C2} = 10.8 \text{ V}$$

The energy stored in each element is

$$W_{L1} = \frac{1}{2}(2 \times 10^{-3})(-1.2)^2 = 1.44 \text{ mJ}$$
$$W_{L2} = \frac{1}{2}(4 \times 10^{-3})(1.8)^2 = 6.48 \text{ mJ}$$
$$W_{C1} = \frac{1}{2}(20 \times 10^{-6})(16.2)^2 = 2.62 \text{ mJ}$$
$$W_{C2} = \frac{1}{2}(50 \times 10^{-6})(10.8)^2 = 2.92 \text{ mJ}$$

Capacitors in Series



Using KVL

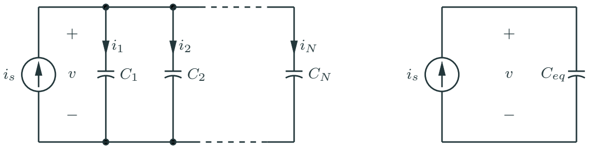
$$v_s = \sum_{n=1}^{N} v_n = \sum_{n=1}^{N} \left[\frac{1}{C_n} \int_{t_0}^{t} i d\tau + v_n(t_0) \right]$$

$$= \left(\sum_{n=1}^{N} \frac{1}{C_n} \right) \int_{t_0}^{t} i d\tau + \sum_{n=1}^{N} v_n(t_0) = \frac{1}{C_{eq}} \int_{t_0}^{t} i d\tau + v_s(t_0)$$

Then

$$C_{eq} = \frac{1}{1/C_1 + 1/C_2 + \cdots + 1/C_N}$$

Capacitors in Parallel



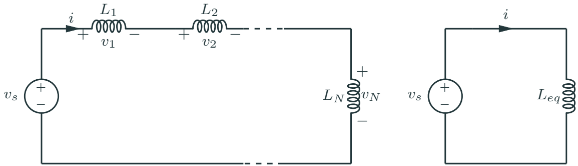
Using KCL

$$
\begin{aligned}
i(t) &= i_1(t) + i_2(t) + \cdots + i_N(t) = C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + \cdots + C_N \frac{dv}{dt} \\
&= \left(\sum_{i=1}^{N} C_i \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}
\end{aligned}
$$

Thus

$$C_{eq} = C_1 + C_2 + \cdots + C_N$$

Inductors in Series



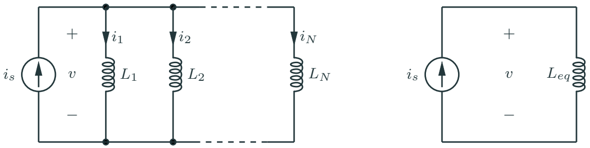
Using KVL

$$
\begin{aligned}
v_s &= \sum_{n=1}^{N} v_n = v_1 + v_2 + \cdots v_N \\
&= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_N \frac{di}{dt} = (L_1 + L_2 + \cdots + L_N) \frac{di}{dt} \\
&= L_{eq} \frac{di}{dt}
\end{aligned}
$$

Then

$$L_{eq} = L_1 + L_2 + \cdots + L_N$$

Inductors in Parallel



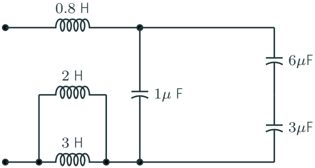
Using KCL

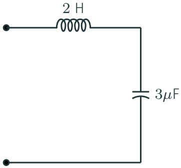
$$
\begin{aligned}
i_s = \sum_{n=1}^{N} i_n &= \sum_{n=1}^{N} \left[\frac{1}{L_n} \int_{t_0}^{t} v(\tau) d\tau + i_n(t_0) \right] \\
&= \left(\sum_{n=1}^{N} \frac{1}{L_n} \right) \int_{t_0}^{t} v(\tau) d\tau + \sum_{n=1}^{N} i_n(t_0) \\
&= \frac{1}{L_{eq}} \int_{t_0}^{t} v(\tau) d\tau + i_s(t_0)
\end{aligned}
$$

Thus

$$
L_{eq} = \frac{1}{1/L_1 + 1/L_2 + \cdots + 1/L_N}
$$

Inductors and Capacitors Combination





We have

$$C_{eq} = 1\ \mu\text{F} + \frac{18}{9}\ \mu\text{F} = 3\ \mu\text{F}$$

$$L_{eq} = 0.8\ \text{H} + \frac{6}{5}\ \text{H} = 2\ \text{H}$$

Reference

1. William H. Hayt, Jr., Jack E. Kemmerly, and Steven M. Durbin *Engineering Circuit Analysis*, 8th Edition McGraw-Hill, 2012.
2. J. David Irwin, and R. Mark Nelms *Basic Engineering Circuit Analysis*, 11th, Wiley, 2015.